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Honorable Mention

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FROM 'CHOPSTICKS' TO PERIODICITY OF GENERALIZED FIBONACCI SEQUENCE

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Abstract. The ultimate objective of this paper is to examine the periodicity of the *Generalized Fibonacci Sequence (GFS)* modulo j with different starting numbers. In this paper, we introduce a brand new method to study the period of the sequence inspired by the hand game 'Chopsticks' usually played in primary schools.

We first prove that the period of *GFS* modulo a prime p other than 5 is either half of the p-th *Pisano Period* or exactly equal to it in **Theorem 16**. We then investigate the decomposition from the period of the game modulo j to the least common multiple of the periods of the game modulo the prime-power factors of j in **Theorem 23**. We continue our investigation on the periodicity of *GFS* modulo p other than 5 and prime powers p^k in **Corollary 18-20, Lemma 7 and Theorem 26**. Finally, we use **Theorem 27** to give a general expression for the period of *GFS* modulo j in terms of the p_i-th *Pisano period*, where p_i's are the prime factors of j.

1. Introduction

The ultimate objective of this paper is to investigate the *generalized Fibonacci sequence* (as defined in [1]) modulo j.

Before studying the periodicity of *generalized Fibonacci sequence* modulo j, we played with a hand game called 'Chopsticks' [4] which is famous among primary school students. The hand game is easy to play and understand. However, it has been very arduous to find the winning strategy We therefore investigate the game step by step.

Firstly, we study the game directly by finding out all possible situations leading to winning the game or giving a draw in the very beginning. We examine the flow of games played by 2 players as a foundation for our further investigation.

We then focus on analyzing repeating strategies to play the game. We discover that some repeating strategies cause different cycles. During investigation of the period of the cycles, we are excited to find that the game involving only one player with two hands tapping each other may have a close relationship with *generalized Fibonacci sequence* modulo j. All the results and conclusions we find about such game are surprisingly correspondent to the periodicity of *generalized Fibonacci sequence* modulo j, with starting numbers **other than** 0,1.

We first examine the period of the game modulo p, a prime, and then generalize the result to power of primes. We further generalize the results to the periodicity of the game modulo j. Finally, we construct our main theorem on the periodicity of *generalized Fibonacci sequence* modulo j using the results in studying the period of the hand game.

2. Hand game "Chopsticks"

2.1. Rules of the Game

To start the game, two players extend one finger on each hand. Players take turn to attack the other. The attacker taps one of his hands against one of the other three.

The number (of fingers) shown by the tapping hand remains unchanged but the number shown by the tapping hand is added to the hand tapped. In other words, the new number shown by the tapped hand is the sum of its own number and the number shown by the tapping hand. If the sum is greater than four, then the number shown by the tapped hand will be subtracted by 5. Mathematically speaking, the game is played under modulo 5. When the number of a hand is ZERO, it will be knocked out of the game. Once the both hands of a player are knocked out, the player loses the game.

2.2. Wise Strategies on Different Games

Firstly, we start to investigate the game involving two players both with 1 hand, and then the game with one player having 2 hands while the other having one hand only. Finally we study the hand game "Chopsticks".

We assume that both players are equally wise to know the right strategies that are beneficial to them. They will first try to win the game, and avoid losing the game if they fail winning.

For the game involving two players having 1 hand, there is no winning strategy can be used for both sides. However, we discover that it is closely related to *Fibonacci sequence* and *Pisano period* (Refer to Appendix A for details). The outcome of the game depends on the initial numbers shown by the hands of the players.

We investigate the game with one player having 2 hands while the other having 1 hand only since we want to know if the player with one hand can win. The investigation seems extremely difficult as there are totally 128 cases. Yet, we can greatly reduce the 128 cases (See Appendix A) by grouping them by the property of module. Finally, we find that under certain initial condition, the player having 2 hands can always win the game.

Ultimately, we can directly investigate the game involving both players having 2 hands. Once again the number of cases can be greatly reduced. We can conclude that both players fail to win the game started at $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}_{\text{mod } 5}$ (See Appendix A). We also try the game with different initial values and find that the outcome of the game depends on the initial values. For example, in Lemma 39 (See Appendix A): Both players fail to win the game at $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}_{\text{mod } 5}$. If we want to investigate the hand game started at $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}_{\text{mod } 5}$, we just need to determine if the game will enter $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}_{\text{mod } 5}$. If so, the game ends with a draw.

2.3. Mechanical Strategies of Hand Games

By repeating mechanical strategies, some cycles are somehow constructed. We hope to investigate the periodicity of the game modulo j under some specific strategies. We have tried all possible mechanical strategies and find that the game goes back to the initial situation eventually. The periodicity of the game modulo j is related to the initial values and the number j. In some cases, the game has a similar behavior with *Fibonacci sequence*. Readers may look up for details of the strategies and results in Appendix B.

3. The Game and the Pisano Period

In the previous investigation, we discover that one of the hand games is closely related to the *Fibonacci sequence*. This is the game restricted to one player with two hands tapping each other turn by turn, i.e. first use one's left hand to tap his right hand, and then use his right hand to tap his left hand, and so on. For simplicity, we call it **the game** throughout the paper. Before exploring the period of the game under modulo j, we need the following definitions.

Definition 1. *The generalized Fibonacci sequence is defined by the recursion*

$$F_n = F_{n-1} + F_{n-2} \quad \textit{for } n \geq 2$$

where $F_0 = a$ and $F_1 = b$ are initial integers.

Definition 2. *Denote by $\pi(a, b, j)$ the period of the generalized Fibonacci sequence modulo j with initial integers a, b.*

Definition 3. $\begin{pmatrix} L_n \\ R_n \end{pmatrix}$ *represents the numbers of both hands of the game after n steps, where L_n and R_n are the numbers of the left hand and right hand after n steps respectively and L_0 and R_0 are the initial integers of the game.*

Definition 4. *For any positive integer n, j, define the n-step game modulo j by*

$$\begin{pmatrix} L_0 & R_0 \\ L_1 & R_1 \\ & \vdots \\ L_n & R_n \end{pmatrix}_{modj} \quad , \textit{where } 0 \leq L_i, R_i < j \ \ \forall i \in \{0, 1, 2, \ldots, n\}$$

Definition 5. *$\sigma(L_0, R_0, j)$ is defined as the period of the game modulo j, which is the smallest positive integer $\sigma(L_0, R_0, j)$ such that*

$$L_{\sigma(L_0,R_0,j)} \equiv L_0 (mod\ j) \ \ \textit{and} \ \ R_{\sigma(L_0,R_0,j)} \equiv R_0 (mod\ j)$$

In most cases, we do not consider $L_0 = R_0 = 0$ as $\sigma(0, 0, j)$ is simply equal to 1.

Now we are going to find the relationship between the periodicity of the game modulo j and that of the *generalized Fibonacci sequence* modulo j in Lemma 6.

Lemma 6.

$$\sigma(L_0, R_0, j) = \begin{cases} \dfrac{\pi(R_0, L_0, j)}{2} & \textit{if } \pi(R_0, L_0, j) \textit{ is even} \\ \pi(R_0, L_0, j) & \textit{if } \pi(R_0, L_0, j) \textit{ is odd} \end{cases}$$

Proof. Rearranging in pairs the terms in the *generalized Fibonacci sequence* modulo j, we can form the array of the game as

$$\begin{pmatrix} F_1 & F_0 \\ F_3 & F_2 \\ F_5 & F_4 \\ & \vdots \\ F_{k+1} & F_k \end{pmatrix}_{\text{mod} j} .$$

Below is the example for $j = 3$ with initial values 0,1.
Generalized Fibonacci sequence modulo 3: $0, 1, 1, 2, 0, 2, 2, 1, 0, 1, \ldots$

The array of the game under modulo 3 is $\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 2 & 0 \\ 1 & 2 \\ 1 & 0 \end{pmatrix}_{\text{mod}3}$.

If the period of *generalized Fibonacci sequence* modulo j is even, then the cycle can be grouped into pairs of two directly. Since the last pair in the cycle is the same with the first, it is obvious that the period of the game is exactly half of the period of *generalized Fibonacci sequence* modulo j.

$$\sigma(L_0, R_0, j) = \frac{\pi(R_0, L_0, j)}{2}$$

On the other hand, it the period of *generalized Fibonacci sequence* modulo j is odd, then the last entry in the cycle of the sequence cannot be paired up. In this case, we require one more cycle to complete the game. In this case, the period of the game is exactly the period of generalized Fibonacci sequence modulo j,

$$\sigma(L_0, R_0, j) = \pi(R_0, L_0, j)$$

Here we show the example when $j = 11$ with initial values $3, 1$.
Generalized Fibonacci sequence modulo 11: $3, 1, 4, 5, 9, 3, 1, \ldots$

The array of the game under modulo 11 is $\begin{pmatrix} 1 & 3 \\ 5 & 4 \\ 3 & 9 \\ 4 & 1 \\ 9 & 5 \\ 1 & 3 \end{pmatrix}_{\text{mod}11}$

In the third and the fourth row of the game, the *generalized Fibonacci sequence* actually comes to the end of a cycle $(3, 1)$, however, they belong to different row. Thus one more cycle is needed for completing a cycle of the game. $\square$

Now we can relate the period of the game modulo j to the *Pisano period* by the following lemmas.

Lemma 7.

$$\sigma(1,0,j) = \begin{cases} \dfrac{\pi(j)}{2} & \textit{when } j > 2 \\ 3 & \textit{when } j = 2 \end{cases}, \textit{ where } \pi(j) \textit{ is the } j\textit{-th Pisano period.}$$

Proof. For $j > 2$, we know that the j-th *Pisano period* $\pi(j)$ is an even number in [3]. Note that the entries of the n-step game modulo j $\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 5 & 3 \\ & \vdots & \\ 1 & 0 \end{pmatrix}_{\text{mod}j}$ form the

Fibonacci sequence modulo j, and it is obvious that the period of the game is half of the j-th *Pisano period* for $j > 2$,

i.e.
$$\sigma(1,0,j) = \frac{\pi(j)}{2} \ \ \forall\, j > 2$$

When $j = 2$, the game becomes $\begin{pmatrix} 1 & 0 \\ 0 & 1 \\ 1 & 1 \\ 1 & 0 \end{pmatrix}_{\text{mod}2}$. Thus $\sigma(1,0,2) = \pi(2) = 3$. □

We are then interested in the game with the initial values multiplied by k and find that the period remains the same in Lemma 8.

Lemma 8.
If $(j,k) = 1$*, then* $\sigma(kL_0, kR_0, j) = \sigma(L_0, R_0, j)$.

Proof.

Consider the game
$$\begin{pmatrix} L_0 & R_0 \\ L_1 & R_1 \\ \vdots & \\ L_n & R_n \\ \vdots & \\ L_{\pi(L_0,R_0,j)} & R_{\pi(L_0,R_0,j)} \end{pmatrix}_{\text{mod}j}$$
and
$$\begin{pmatrix} kL_0 & kR_0 \\ kL_1 & kR_1 \\ \vdots & \\ kL_n & kR_n \\ \vdots & \\ kL_{\pi(L_0,R_0,j)} & kR_{\pi(L_0,R_0,j)} \end{pmatrix}_{\text{mod}j}.$$

Suppose $(j,k) = 1$, then we have
$$\begin{cases} L_n \neq L_0 \\ R_n \neq R_0 \end{cases} \pmod{j} \Rightarrow \begin{cases} kL_n \neq kL_0 \\ kR_n \neq kR_0 \end{cases} \pmod{j} \quad \forall\, 0 < n < \pi(L_0,R_0,j)$$
But by definition, $\begin{cases} L_{\sigma(L_0,R_0,j)} \equiv L_0 \\ R_{\sigma(L_0,R_0,j)} \equiv R_0 \end{cases} \pmod{j}$, thus
$$\begin{cases} kL_{\sigma(L_0,R_0,j)} \equiv kL_0 \\ kR_{\sigma(L_0,R_0,j)} \equiv kR_0 \end{cases} \pmod{j},$$

i.e.
$$\sigma(kL_0, kR_0, j) = \sigma(L_0, R_0, j)$$

Therefore,
$$(j,k) = 1 \ \Rightarrow\ \sigma(kL_0, kR_0, j) = \sigma(L_0, R_0, j)$$
□

Then we investigate the game when one of the initial numbers is zero and find that the period of the game is just the same as that with initial values 0,1 or 1,0 in Lemma 9

Lemma 9. *For any L_0, R_0 and j such that $(L_0, j) = (R_0, j) = 1$, the games* $\begin{pmatrix} L_0 & 0 \\ 2L_0 & L_0 \\ \vdots & \\ L_0 & 0 \end{pmatrix}_{modj}$ *and* $\begin{pmatrix} 0 & R_0 \\ R_0 & R_0 \\ \vdots & \\ 0 & R_0 \end{pmatrix}_{modj}$ *have same period for all positive L_0 and R_0.*

i.e.
$$\sigma(L_0, 0, j) = \sigma(0, R_0, j) = \begin{cases} \dfrac{\pi(j)}{2} & \textit{when } j > 2 \\ 3 & \textit{when } j = 2 \end{cases}$$

Proof. Obviously, $\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 5 & 3 \\ \vdots & \\ 1 & 0 \end{pmatrix}_{\text{mod}j}$ and $\begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 2 \\ \vdots & \\ 0 & 1 \end{pmatrix}_{\text{mod}j}$ have same period as they both perform in similarly as the j-th *Pisano cycle*, we thus have $\sigma(1, 0, j) = \sigma(0, 1, j)$. By Lemma 7 and Lemma 8, if $(L_0, j) = (R_0, j) = 1$, we get

$$\sigma(L_0, 0, j) = \sigma(1, 0, j) = \sigma(0, 1, j) = \sigma(0, R_0, j) = \begin{cases} \dfrac{\pi(j)}{2} & \text{when } j > 2 \\ 3 & \text{when } j = 2 \end{cases}$$

□

Now we are going to link up the game and *Fibonacci sequence* by successfully express the L_n and R_n in terms of *Fibonacci number* and the initial values of the game in **Theorem 10**.

Theorem 10. $\begin{pmatrix} L_n \\ R_n \end{pmatrix} = \begin{pmatrix} F_{2n+1}L_0 + F_{2n}R_0 \\ F_{2n}L_0 + F_{2n-1}R_0 \end{pmatrix}$, *where F_n is the nth Fibonacci number with $F_0 = 0$ and $F_1 = F_2 = 1$.*

Proof 1: Magically, it can be easily proved by combining two independent games to form another game as below:

$$\begin{pmatrix} L_0 & 0 \\ 2L_0 & L_0 \\ \vdots & \\ L_0F_{2n+1} & L_0F_{2n} \end{pmatrix}_{\text{mod}j} + \begin{pmatrix} 0 & R_0 \\ R_0 & R_0 \\ \vdots & \\ R_0F_{2n} & R_0F_{2n-1} \end{pmatrix}_{\text{mod}j}$$

$$= \begin{pmatrix} L_0 & R_0 \\ 2L_0 + R_0 & L_0 + R_0 \\ \vdots & \\ L_0F_{2n+1} + R_0F_{2n} & L_0F_{2n} + R_0F_{2n-1} \end{pmatrix}_{\bmod j}.$$

Therefore, $\begin{pmatrix} L_n \\ R_n \end{pmatrix} = \begin{pmatrix} L_0F_{2n+1} + R_0F_{2n} \\ L_0F_{2n} + R_0F_{2n-1} \end{pmatrix}$ □

However, we do want to know the general terms for the game with initial values L_0 and R_0 and n only. If the step of the game can be treated as a transformation, we may use the method of diagonalization to find its power and hence the general term of the game.

To start, we have $\begin{pmatrix} L_1 \\ R_1 \end{pmatrix} = \begin{pmatrix} 2L_0 + R_0 \\ L_0 + R_0 \end{pmatrix}$, where L_0 and R_0 are the initial values of the left hand and right hand respectively. Thus each step can be represented by the transformation matrix $T = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}$. Now we have to compute T^n in order to find the general form of $\begin{pmatrix} L_n \\ R_n \end{pmatrix}$. For convenience, it is assumed that the player can show any number by each of his both hands.

Theorem 11.

$$T^n = \frac{1}{\sqrt{5}} \begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{2n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n+1} & \left(\frac{1+\sqrt{5}}{2}\right)^{2n} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n} \\ \left(\frac{1+\sqrt{5}}{2}\right)^{2n} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n} & \left(\frac{1+\sqrt{5}}{2}\right)^{2n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n-1} \end{pmatrix}$$

Proof. Suppose $\det(T - \lambda I)$=$\det\begin{pmatrix} 2-\lambda & 1 \\ 1 & 1-\lambda \end{pmatrix} = 0$ for some real number λ.
Then we have $(2-\lambda)(1-\lambda) - 1 = \lambda^2 - 3\lambda + 1 = 0$ which gives the eigenvalues of T, i.e. $\lambda = \dfrac{3 \pm \sqrt{5}}{2}$.
When $\lambda = \dfrac{3+\sqrt{5}}{2}$, by considering $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \lambda\begin{pmatrix} x_1 \\ y_1 \end{pmatrix} = \dfrac{3+\sqrt{5}}{2}\begin{pmatrix} x_1 \\ y_1 \end{pmatrix}$, we have

$$\begin{cases} 2x_1 + y_1 = \dfrac{3+\sqrt{5}}{2}x_1 \\ x_1 + y_1 = \dfrac{3+\sqrt{5}}{2}y_1 \end{cases} \Rightarrow x_1 : y_1 = \frac{1+\sqrt{5}}{2}$$

Thus, $\begin{pmatrix} 1+\sqrt{5} \\ 2 \end{pmatrix}$ can be an eigenvector for the eigenvalue $\lambda = \dfrac{3+\sqrt{5}}{2}$.
When $\lambda = \dfrac{3-\sqrt{5}}{2}$, by considering $\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \lambda\begin{pmatrix} x_2 \\ y_2 \end{pmatrix} = \dfrac{3-\sqrt{5}}{2}\begin{pmatrix} x_2 \\ y_2 \end{pmatrix}$, we

have

$$\begin{cases} 2x_2 + y_2 = \dfrac{3-\sqrt{5}}{2}x_2 \\ x_2 + y_2 = \dfrac{3-\sqrt{5}}{2}y_2 \end{cases} \Rightarrow x_2 : y_2 = \frac{1-\sqrt{5}}{2}$$

Thus, $\begin{pmatrix} 1-\sqrt{5} \\ 2 \end{pmatrix}$ can be an eigenvector for the eigenvalue $\lambda = \dfrac{3-\sqrt{5}}{2}$.

Let $Q = \begin{pmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{pmatrix}$, then $Q^{-1} = \frac{1}{4\sqrt{5}} \begin{pmatrix} 2 & -1+\sqrt{5} \\ -2 & 1+\sqrt{5} \end{pmatrix}$ and we have

$$\begin{aligned} Q^{-1}TQ &= \frac{1}{4\sqrt{5}} \begin{pmatrix} 2 & -1+\sqrt{5} \\ -2 & 1+\sqrt{5} \end{pmatrix} \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix} \begin{pmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{3+\sqrt{5}}{2} & 0 \\ 0 & \dfrac{3-\sqrt{5}}{2} \end{pmatrix} \\ &= \begin{pmatrix} \dfrac{1+\sqrt{5}}{2} & 0 \\ 0 & \dfrac{1-\sqrt{5}}{2} \end{pmatrix}^2 \end{aligned}$$

Then

$$\begin{aligned} T^n &= Q(Q^{-1}T_{2\times 2}Q)^n Q^{-1} \\ &= \frac{1}{4\sqrt{5}} \begin{pmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{pmatrix} \begin{pmatrix} \dfrac{1+\sqrt{5}}{2} & 0 \\ 0 & \dfrac{1-\sqrt{5}}{2} \end{pmatrix}^{2n} \begin{pmatrix} 2 & -1+\sqrt{5} \\ -2 & 1+\sqrt{5} \end{pmatrix} \end{aligned}$$

$$\begin{aligned} T^n &= \frac{1}{2^{2n+2}\sqrt{5}} \begin{pmatrix} 1+\sqrt{5} & 1-\sqrt{5} \\ 2 & 2 \end{pmatrix} \begin{pmatrix} 2(1+\sqrt{5})^{2n} & (-1+\sqrt{5})(1+\sqrt{5})^{2n} \\ -2(1-\sqrt{5})^{2n} & (1+\sqrt{5})(1-\sqrt{5})^{2n} \end{pmatrix} \\ &= \frac{1}{2^{2n+2}\sqrt{5}} \begin{pmatrix} 2(1+\sqrt{5})^{2n+1} - 2(1-\sqrt{5})^{2n+1} & 4(1+\sqrt{5})^{2n} - 4(1-\sqrt{5})^{2n} \\ 4(1+\sqrt{5})^{2n} - 4(1-\sqrt{5})^{2n} & 8(1+\sqrt{5})^{2n-1} - 8(1-\sqrt{5})^{2n-1} \end{pmatrix} \\ &= \frac{1}{\sqrt{5}} \begin{pmatrix} \left(\dfrac{1+\sqrt{5}}{2}\right)^{2n+1} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{2n+1} & \left(\dfrac{1+\sqrt{5}}{2}\right)^{2n} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{2n} \\ \left(\dfrac{1+\sqrt{5}}{2}\right)^{2n} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{2n} & \left(\dfrac{1+\sqrt{5}}{2}\right)^{2n-1} - \left(\dfrac{1-\sqrt{5}}{2}\right)^{2n-1} \end{pmatrix} \end{aligned}$$

□

Using the general term for the *Fibonacci sequence* and Theorem 11, we can give another proof for the Theorem 10.

Proof 2 for Theorem 10: By Theorem 11, we have

$$T^n = \frac{1}{\sqrt{5}}\begin{pmatrix} \left(\frac{1+\sqrt{5}}{2}\right)^{2n+1} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n+1} & \left(\frac{1+\sqrt{5}}{2}\right)^{2n} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n} \\ \left(\frac{1+\sqrt{5}}{2}\right)^{2n} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n} & \left(\frac{1+\sqrt{5}}{2}\right)^{2n-1} - \left(\frac{1-\sqrt{5}}{2}\right)^{2n-1} \end{pmatrix}$$
$$= \begin{pmatrix} F_{2n+1} & F_{2n} \\ F_{2n} & F_{2n-1} \end{pmatrix}$$

Therefore, $\begin{pmatrix} L_n \\ R_n \end{pmatrix} = T^n \begin{pmatrix} L_0 \\ R_0 \end{pmatrix} = \begin{pmatrix} F_{2n+1} & F_{2n} \\ F_{2n} & F_{2n-1} \end{pmatrix}\begin{pmatrix} L_0 \\ R_0 \end{pmatrix} = \begin{pmatrix} F_{2n+1}L_0 + F_{2n}R_0 \\ F_{2n}L_0 + F_{2n-1}R_0 \end{pmatrix}$ □

By Theorem 10, we get the following Identity 1.

Identity 12. $F_{2n-1}F_{2n+1} - {F_{2n}}^2 = 1$

Proof. By Theorem 11, we have $\begin{pmatrix} F_{2n+1} & F_{2n} \\ F_{2n} & F_{2n-1} \end{pmatrix} = \begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}^n$
Taking the determinant of both sides, we get

$$F_{2n+1}F_{2n-1} - {F_{2n}}^2 = \left[\det\begin{pmatrix} 2 & 1 \\ 1 & 1 \end{pmatrix}\right]^n = 1^n = 1$$

□

Identity 13. $(F_{n+1} + F_{n-1})^2 = L_n^2 = 5{F_n}^2 + 4(-1)^n$ *where* L_n *is the* n^{th} *Lucas number.*

Proof. Let $\phi = \dfrac{1+\sqrt{5}}{2}$ and $\bar{\phi} = \dfrac{1-\sqrt{5}}{2}$.
The general forms of *Fibonacci number* and *Lucas number* are known as follow:

$$F_n = \frac{\phi^n - \bar{\phi}^n}{\sqrt{5}} \quad \text{and} \quad L_n = \phi^n + \bar{\phi}^n$$

Now we have

$$\begin{aligned} F_{n+1} + F_{n-1} &= \frac{\phi^{n+1} - \bar{\phi}^{n+1}}{\sqrt{5}} + \frac{\phi^{n-1} - \bar{\phi}^{n-1}}{\sqrt{5}} \\ &= \frac{\phi^n(\phi - \phi^{-1}) + \bar{\phi}^n(-\bar{\phi}^{-1} - \bar{\phi})}{\sqrt{5}} \end{aligned}$$

But we have

$$\begin{aligned} \phi^{-1} &= \frac{2}{1+\sqrt{5}} = \frac{2(1-\sqrt{5})}{-4} = -\left(\frac{1-\sqrt{5}}{2}\right) = -\bar{\phi} \\ \bar{\phi}^{-1} &= \frac{2}{1-\sqrt{5}} = \frac{2(1+\sqrt{5})}{-4} = -\left(\frac{1+\sqrt{5}}{2}\right) = -\phi \end{aligned}$$

Then

$$\begin{aligned} F_{n+1}+F_{n-1} &= \frac{\phi^n(\phi-\phi^{-1})+\bar{\phi}^n(-\bar{\phi}^{-1}-\bar{\phi})}{\sqrt{5}} \\ &= \frac{\phi^n(\phi-\bar{\phi})+\bar{\phi}^n(\phi-\bar{\phi})}{\sqrt{5}} \\ &= (\phi^n+\bar{\phi}^n)F_1 \\ &= \phi^n+\bar{\phi}^n = L_n \end{aligned}$$

Now since

$$\begin{aligned} (F_{n+1}+F_{n-1})^2-5{F_n}^2-4(-1)^n &= (\phi^n+\bar{\phi}^n)^2-5\Big(\frac{\phi^n-\bar{\phi}^n}{\sqrt{5}}\Big)^2-4(-1)^n \\ &= (\phi^n+\bar{\phi}^n)^2-(\phi^n-\bar{\phi}^n)^2-4(-1)^n \\ &= (2\phi^n)(2\bar{\phi}^n)-4(-1)^n \\ &= 4\Big[\Big(\frac{1+\sqrt{5}}{2}\Big)\Big(\frac{1-\sqrt{5}}{2}\Big)\Big]^n-4(-1)^n \\ &= 4(-1)^n-4(-1)^n=0 \end{aligned}$$

Thus we have $(F_{n+1}+F_{n-1})^2={L_n}^2=5{F_n}^2+4(-1)^n$. □

Theorem 14. *Let p be any prime number other than 5. The period $\sigma(L_0,R_0,p)$ of the game starting with L_0 and R_0, is* ***independent*** *of L_0 and R_0, where L_0 and R_0 are not both zero.*

Proof. For simplicity, we let $x=L_0$ and $y=R_0$.

Case I: either $x=0$ or $y=0$

By lemma 9, $\sigma(x,0,p)=\sigma(0,y,p)=\begin{cases} \dfrac{\pi(p)}{2} & \text{when } p>2 \\ 3 & \text{when } p=2 \end{cases}$

Case II: $x\neq 0 \wedge y\neq 0$

Note that we can combine two independent games starting with $x,0$ and $0,y$ to form a new game with initial values x and y.

Since $\sigma(x,0,p)=\sigma(0,y,p)\ \forall\ 0<x,y<p$, we have

$$\begin{pmatrix} x & 0 \\ 2x & x \\ \vdots & \\ xF_{2n+1} & xF_{2n} \\ \vdots & \\ x & 0 \end{pmatrix}_{\bmod p} + \begin{pmatrix} 0 & y \\ y & y \\ \vdots & \\ yF_{2n} & yF_{2n-1} \\ \vdots & \\ 0 & y \end{pmatrix}_{\bmod p}$$

$$= \begin{pmatrix} x & y \\ 2x+y & x+y \\ & \vdots \\ xF_{2n+1}+yF_{2n} & xF_{2n}+yF_{2n-1} \\ & \vdots \\ x & y \end{pmatrix}_{\text{mod} p}.$$

From the equation, it is obvious that $\sigma(x,y,p) \mid \sigma(x,0,p)$ and $\sigma(x,y,p) \mid \sigma(0,y,p)$. Consider

$$\begin{cases} xF_{2n+1}+yF_{2n} \equiv x \pmod p & \text{(i)} \\ xF_{2n}+yF_{2n-1} \equiv y \pmod p & \text{(ii)} \end{cases}$$

From (i) & (ii),

$$F_{2n} \equiv 0 \pmod p \Leftrightarrow F_{2n+1} \equiv F_{2n-1} \equiv 1 \pmod p$$

But $(x \quad 0)$ should appear ONLY in the first and last rows of $\begin{pmatrix} x & 0 \\ 2x & x \\ & \vdots \\ xF_{2n+1} & xF_{2n} \\ & \vdots \\ x & 0 \end{pmatrix}_{\text{mod } p}$.

$$\therefore F_{2n} \equiv 0 \pmod p \Rightarrow n = \sigma(x,0,p) \text{ or } n = 0$$

i.e. $$F_{2n} \equiv 0 \pmod p \Rightarrow \sigma(x,y,p) = \sigma(x,0,p) = \sigma(0,y,p) \qquad (*)$$

Now assume $F_{2n} \not\equiv 0 \pmod p$, (i) and (ii) can be transformed to

$$\begin{cases} yF_{2n}{}^2 \equiv (xF_{2n})(1-F_{2n+1}) \pmod p & \text{(iii)} \\ xF_{2n} \equiv y(1-F_{2n-1}) \pmod p & \text{(iv)} \end{cases}$$

Substituting (iv) into (iii), we have

$$yF_{2n}{}^2 \equiv y(1-F_{2n+1}-F_{2n-1}+F_{2n+1}F_{2n-1}) \pmod p$$
$$F_{2n+1}+F_{2n-1} \equiv F_{2n+1}F_{2n-1}-F_{2n}{}^2+1 \pmod p$$

By Identity 12, we have $F_{2n+1}F_{2n-1}-F_{2n}{}^2=1$ and hence $F_{2n+1}+F_{2n-1} \equiv 2 \pmod p$.
Then $(F_{2n+1}+F_{2n-1})^2 \equiv 4 \pmod p$.
By Identity 13, we have $(F_{2n+1}+F_{2n-1})^2 = 5F_{2n}{}^2+4(-1)^{2n} = 5F_{2n}{}^2+4$.

Thus $5{F_{2n}}^2 \equiv 0 \pmod{p}$.
But $F_{2n} \not\equiv 0 \pmod{p}$, therefore $5 \equiv 0 \pmod{p}$, i.e. $p = 5$

$$\therefore \qquad F_{2n} \not\equiv 0 (\text{mod } p) \Rightarrow p = 5$$
$$\text{i.e.} \qquad p \neq 5 \Rightarrow F_{2n} \equiv 0 \pmod{p}$$

Combining (*), we have

$$p \neq 5 \Rightarrow \sigma(x, y, p) = \sigma(x, 0, p) = \sigma(0, y, p) = \begin{cases} \dfrac{\pi(p)}{2} & \text{when } p > 2 \\ 3 & \text{when } p = 2 \end{cases}$$

□

For completeness, we will study the case $p = 5$ in the Lemma 15.

Lemma 15.

$$\sigma(L_0, R_0, 5) = \begin{cases} 2 & \textit{if } {R_0}^2 - {L_0}^2 + L_0R_0 \equiv 0 \textit{ (mod 5)} \\ 10 & \textit{otherwise} \end{cases}$$

Proof. Recall the simultaneous equation in Theorem 10. Let $x = L_0, y = R_0$ and $p = 5$.

$$\begin{cases} xF_{2n+1} + yF_{2n} \equiv x \text{ (mod5)} & \text{(i)} \\ xF_{2n} + yF_{2n-1} \equiv y \text{ (mod5)} & \text{(ii)} \end{cases}$$

Note that $(x, p) = (y, p) = 1$, we can compute (i) $\times y -$ (ii) $\times x$:

$$(y^2 - x^2)F_{2n} + xy(F_{2n+1} - F_{2n-1}) \equiv 0 \text{ (mod5)}$$
$$(y^2 - x^2 + xy)F_{2n} \equiv 0 \text{ (mod5)} \qquad \text{(v)}$$

The solutions of $y^2 - x^2 + xy \equiv 0$ (mod5) for $0 < x, y < 5$ are

$$\begin{cases} x = 3 \\ y = 1 \end{cases} \quad \begin{cases} x = 2 \\ y = 4 \end{cases} \quad \begin{cases} x = 1 \\ y = 2 \end{cases} \quad \begin{cases} x = 4 \\ y = 3 \end{cases}$$

For $p = 5$, it is easy to exhaust all cases. All games are listed below.

$$\begin{pmatrix} 1 & 0 \\ 2 & 1 \\ 0 & 3 \\ 3 & 3 \\ 4 & 1 \\ 4 & 0 \\ 3 & 4 \\ 0 & 2 \\ 2 & 2 \\ 1 & 4 \\ 1 & 0 \end{pmatrix}_{\text{mod5}} \quad \begin{pmatrix} 0 & 1 \\ 1 & 1 \\ 3 & 2 \\ 3 & 0 \\ 1 & 3 \\ 0 & 4 \\ 4 & 4 \\ 2 & 3 \\ 2 & 0 \\ 4 & 2 \\ 0 & 1 \end{pmatrix}_{\text{mod5}} \quad \begin{pmatrix} 3 & 1 \\ 2 & 4 \\ 3 & 1 \end{pmatrix}_{\text{mod5}} \quad \begin{pmatrix} 1 & 2 \\ 4 & 3 \\ 1 & 2 \end{pmatrix}_{\text{mod5}}$$

From the above cases, we can conclude that $\sigma(x, y, 5) = 2$ only for these choice of x and y, then we have

$$\sigma(x, y, 5) = 2 \Leftrightarrow y^2 - x^2 + xy \equiv 0 \pmod 5$$

For other choice of x and y, $\sigma(x, y, 5) = \dfrac{\pi(5)}{2} = 10$. □

By Theorem 14 and Lemma 15, we can made a conclusion that for any prime number p,

either $$p \neq 5, \sigma(L_0, R_0, p) = \begin{cases} \dfrac{\pi(p)}{2} & \text{when } p > 2 \\ 3 & \text{when } p = 2 \end{cases},$$

or $$p = 5, \sigma(L_0, R_0, 5) = \begin{cases} 2 & \text{if } {R_0}^2 - {L_0}^2 + L_0 R_0 \equiv 0 \text{ (mod5)} \\ 10 & \text{otherwise} \end{cases}$$

4. Period of generalized Fibonacci sequence modulo p

We completely finish the investigation on the period of the game modulo p for all prime numbers p. The periods are independent of the initial value of both hands and are either constants 2, 3 and 10 or a function of p. By combining Lemma 6 and Theorem 14, we obtain our Main Theorem 16.

Main Theorem 16. *Let $p \neq 5$ be a prime. The period of generalized Fibonacci sequence modulo p, $\pi(a, b, p)$ is either $\pi(p)$ or $\frac{\pi(p)}{2}$, where $\pi(p)$ is the p-th Pisano Period.*

Proof. Let $p \neq 5$ be a prime.
By Lemma 6,

$$\sigma(L_0, R_0, p) = \begin{cases} \dfrac{\pi(R_0, L_0, p)}{2} & \text{if } \pi(R_0, L_0, p) \text{ is even} \\ \pi(R_0, L_0, p) & \text{if } \pi(R_0, L_0, p) \text{ is odd} \end{cases}$$

By Theorem 14,

$$p \neq 5, \sigma(L_0, R_0, p) = \begin{cases} \dfrac{\pi(p)}{2} & \text{when } p > 2 \\ 3 & \text{when } p = 2 \end{cases}$$

Notice that $\pi(2) = 3$. Using the above results, we may conclude that

$$\pi(R_0, L_0, p) = \begin{cases} \dfrac{\pi(p)}{2} & \text{if } p > 2 \text{ and } \pi(R_0, L_0, p) \text{ is odd} \\ 2\pi(p) & \text{if } p = 2 \text{ and } \pi(R_0, L_0, p) \text{ is even} \\ \pi(p) & \text{otherwise} \end{cases}$$

However, we can easily check that $\pi(R_0, L_0, 2)$ must be odd since

$$\pi(1, 0, 2) = \pi(1, 1, 2) = \pi(0, 1, 2) = \pi(2) = 3$$

Therefore, the only possible value $\pi(R_0, L_0, p)$ is $\pi(p)$ or $\dfrac{\pi(p)}{2}$ □

We denote $\pi(a, b, p)$ be π_p in the later sections for simplicity.

Although we can tell the period of *generalized Fibonacci sequence* modulo p other than 5, π_p is either half of the period of the *Fibonacci sequence* modulo p or exactly equal to it, we are still interested in knowing the divisibility of π_p using the periodicity of the game.

Theorem 17. *Let p be a prime. If $p \equiv \pm 1$ (mod5), then $\sigma(L_0, R_0, p) \mid \dfrac{p-1}{2}$.*

Proof. Let p be a prime and $p \equiv \pm 1$ (mod5).
There are some relations between Fibonacci number and Legendre symbol [5].

$$F_{p-\left(\frac{p}{5}\right)} \equiv 0 \ (\mathrm{mod}\ p) \quad \text{and} \quad F_p \equiv \left(\frac{p}{5}\right) \ (\mathrm{mod}\ p)$$

Note that we have the following.

$$\left(\frac{p}{5}\right) = \begin{cases} \left(\frac{1}{5}\right) \equiv 1^{\frac{5-1}{2}} \equiv 1 \ (\mathrm{mod}5) & \Leftrightarrow \quad p \equiv 1 \ (\mathrm{mod}5) \\ \left(\frac{2}{5}\right) \equiv 2^{\frac{5-1}{2}} \equiv -1 \ (\mathrm{mod}5) & \Leftrightarrow \quad p \equiv 2 \ (\mathrm{mod}5) \\ \left(\frac{3}{5}\right) \equiv 3^{\frac{5-1}{2}} \equiv -1 \ (\mathrm{mod}5) & \Leftrightarrow \quad p \equiv 3 \ (\mathrm{mod}5) \\ \left(\frac{4}{5}\right) \equiv 4^{\frac{5-1}{2}} \equiv 1 \ (\mathrm{mod}5) & \Leftrightarrow \quad p \equiv 4 \ (\mathrm{mod}5) \end{cases}$$

That is
$$\left(\frac{p}{5}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \ (\mathrm{mod}5) \\ -1 & \text{if } p \equiv \pm 2 \ (\mathrm{mod}5) \end{cases}$$

If $p \equiv \pm 1$ (mod5), we have

$$F_{p-1} \equiv 0 \ (\mathrm{mod}p) \quad \text{and} \quad F_p \equiv 1 \ (\mathrm{mod}p)$$

Then

$$F_{p-2} = F_p - F_{p-1} \equiv 1 - 0 \equiv 1 \ (\mathrm{mod}p)$$

By Theorem 10, we have
$$\begin{cases} L_n = F_{2n+1}L_0 + F_{2n}R_0 \\ R_n = F_{2n}L_0 + F_{2n-1}R_0 \end{cases}$$

Putting $n = \frac{p-1}{2}$,

$$\begin{cases} L_{\frac{p-1}{2}} = F_{2\left(\frac{p-1}{2}\right)+1}L_0 + F_{2\left(\frac{p-1}{2}\right)}R_0 \equiv F_pL_0 + F_{p-1}R_0 \equiv L_0 & (\mathrm{mod}p) \\ R_{\frac{p-1}{2}} = F_{2\left(\frac{p-1}{2}\right)}L_0 + F_{2\left(\frac{p-1}{2}\right)-1}R_0 \equiv F_{p-1}L_0 + F_{p-2}R_0 \equiv R_0 & (\mathrm{mod}p) \end{cases}$$

Thus $\sigma(L_0, R_0, p) \mid \dfrac{p-1}{2}$ for $p \equiv \pm 1$ (mod5). □

Corollary 18. *If p is a prime number satisfying $p \equiv \pm 1$ (mod5), then $\pi_p \mid p - 1$.*

Proof. By Theorem 16 and Theorem 17, the statement is proved. □

Theorem 19. *If p is a prime number satisfying $p \equiv \pm 2$ (mod5), then $\sigma(L_0, R_0, p) \mid p+1$ and $\frac{p+1}{\sigma(L_0,R_0,p)}$ is odd.*

Proof. Let p be aprime and $p \equiv \pm 2$ (mod5).
By Theorem 17, we have $F_{p-\left(\frac{p}{5}\right)} \equiv 0$ (mod$p$) and $F_{\equiv}\left(\frac{p}{5}\right)$ (mod$p$)

$$\left(\frac{p}{5}\right) = \begin{cases} 1 & \text{if } p \equiv \pm 1 \text{ (mod5)} \\ -1 & \text{if } p \equiv \pm 2 \text{ (mod5)} \end{cases}$$

Now since $p \equiv \pm 2$ (mod5), we have

$$F_{p+1} \equiv 0 \text{ (mod}p) \quad \text{and} \qquad F_p \equiv -1 \text{ (mod}p)$$

Then

$$F_{p+2} = F_p + F_{p+1} \equiv -1 + 0 \equiv -1 \text{ (mod}p)$$

By Lemma 15, we have $\begin{cases} L_n = F_{2n+1}L_0 + F_{2n}R_0 \\ R_n = F_{2n}L_0 + F_{2n-1}R_0 \end{cases}$

Putting $n = \dfrac{p+1}{2}$, we get

$$\begin{cases} L_{\frac{p+1}{2}} = F_{2(\frac{p+1}{2})+1}L_0 + F_{2(\frac{p+1}{2})}R_0 \equiv F_{p+2}L_0 + F_{p+1}R_0 \equiv -L_0 & \text{(mod}p) \\ L_{\frac{p+1}{2}} = F_{2(\frac{p+1}{2})}L_0 + F_{2(\frac{p+1}{2})-1}R_0 \equiv F_{p+1}L_0 + F_pR_0 \equiv -R_0 & \text{(mod}p) \end{cases}$$

Therefore the period does not divide $\dfrac{p+1}{2}$ for $p \equiv \pm 2$ (mod5).

From the above, we know that $\dfrac{p+1}{2}$ steps after $\begin{pmatrix} L_0 \\ R_0 \end{pmatrix}$ is $\begin{pmatrix} -L_0 \\ -R_0 \end{pmatrix}$. Now consider another $\dfrac{p+1}{2}$ steps later, that is $p+1$ steps after $\begin{pmatrix} L_0 \\ R_0 \end{pmatrix}$.

$$\begin{cases} L_{p+1} = F_{2(\frac{p+1}{2})+1}(-L_0) + F_{2(\frac{p+1}{2})}(-R_0) \equiv F_{p+2}(-L_0) + F_{p+1}(-R_0) \equiv L_0 & \text{(mod}p) \\ R_{p+1} = F_{2(\frac{p+1}{2})}(-L_0) + F_{2(\frac{p+1}{2})-1}(-R_0) \equiv F_{p+1}(-L_0) + F_p(-R_0) \equiv R_0 & \text{(mod}p) \end{cases}$$

Thus if $p \equiv \pm 2$ (mod5), we have $\sigma(L_0, R_0, p) \mid p+1$ with $\dfrac{p+1}{\sigma(L_0, R_0, p)}$ odd. □

Corollary 20. *If p is a prime number satisfying $p \equiv \pm 2$ (mod5), then $\pi_p \mid 2p+2$.*

Proof. By Theorem 16, we have
For $p \neq 2,\ \pi_p = \pi(p) = 2\sigma(L_0, R_0, p)$.
For $p = 2,\ \pi_2 = \pi(2) = \sigma(L_0, R_0, 2) = 3$.
Since $\sigma(L_0, R_0, 2) \mid p+1$ by Theorem 19, $\sigma(L_0, R_0, 2) \mid 2p+2$ and then $\pi_2 \mid 2p+2$.

Conclusively, for any prime number p other than 5, the divisibility of π_p is listed as below:

$$\begin{cases} \pi_p \mid p-1 \text{ if } p \equiv \pm 1 \pmod{5} \\ \pi_p \mid 2p+2 \text{ if } p \equiv \pm 2 \pmod{5} \end{cases}$$

□

5. Period of the game modulo a product of relatively prime numbers $n_1 n_2$

After studying the properties of the periodicity of *generalized Fibonacci sequence* modulo a prime p using the period $\sigma(L_0, R_0, p)$ of the game, we further study the game modulo a product $n_1 n_2$ of two relatively prime numbers n_1 and n_2. We successfully find a beautiful formula for decomposing the period of game modulo a product of two relatively-prime integers in Corollary 22 into the L.C.M. of their own periods.

Theorem 21. *For any two numbers n_1 and n_2, if* $\begin{cases} L_0 \equiv {}^1L_0 & (\text{mod } n_1) \\ R_0 \equiv {}^1R_0 & (\text{mod } n_1) \end{cases}$ *and* $\begin{cases} L_0 \equiv {}^2L_0 & (\text{mod } n_2) \\ R_0 \equiv {}^2R_0 & (\text{mod } n_2) \end{cases}$, *we have*

$$\sigma(L_0, R_0, [n_1, n_2]) = [\sigma({}^1L_0, {}^1R_0, n_1), \sigma({}^2L_0, {}^2R_0, n_2)].$$

Proof. Let $m_i = \sigma({}^iL_0, {}^iR_0, n_i)$ for $i = 1, 2$. Consequently, we have

$$\begin{cases} {}^iL_{m_i} \equiv {}^iL_0 & (\text{mod } n_i) \\ {}^iR_{m_i} \equiv {}^iR_0 & (\text{mod } n_i) \end{cases}$$

By the property of a period, it is trivial that for all positive integer r_i,

$$\begin{cases} {}^iL_{r_i m_i} \equiv {}^iL_0 & (\text{mod } n_i) \\ {}^iR_{r_i m_i} \equiv {}^iR_0 & (\text{mod } n_i) \end{cases}$$

Now we consider for some positive integer M,

$$\begin{cases} {}^iL_M \equiv {}^iL_0 & (\text{mod } n_i) \\ {}^iR_M \equiv {}^iR_0 & (\text{mod } n_i) \end{cases}$$

The smallest integer M satisfying these equations is simply the least common multiple of m_1 and m_2, i.e. $[m_1, m_2]$. In this case $r_i = \dfrac{[m_1, m_2]}{m_i}$.

Now we let $M = [m_1, m_2]$ and $N = [n_1, n_2]$.

$$\begin{cases} {}^iL_M \equiv {}^iL_0 & (\text{mod } n_i) \\ {}^iR_M \equiv {}^iR_0 & (\text{mod } n_i) \end{cases} \Rightarrow \begin{cases} {}^iL_M \equiv {}^iL_0 + Nx \\ {}^iR_M \equiv {}^iR_0 + Ny \end{cases} \text{for some } x, y \in Z$$

Then we have M being the smallest integer such that

$$\begin{cases} L_M \equiv F_{2M+1}L_0 + F_{2M}R_0 \equiv F_{2M+1}{}^iL_0 + F_{2M}{}^iR_0 \equiv {}^iL_M \equiv L_0 & (\text{mod } N) \\ R_M \equiv F_{2M}L_0 + F_{2M-1}R_0 \equiv F_{2M}{}^iL_0 + F_{2M-1}{}^iR_0 \equiv {}^iR_M \equiv R_0 & (\text{mod } N) \end{cases}$$

Thus $\sigma(L_0, R_0, N) = M = [\sigma({}^1L_0, {}^1R_0, n_1), \sigma({}^2L_0, {}^2R_0, n_2)]$ □

Corollary 22. *For any two relatively prime numbers n_1 and n_2,*

$$\sigma(L_0, R_0, n_1 n_2) = [\sigma({}^1L_0, {}^1R_0, n_1), \sigma({}^2L_0, {}^2R_0, n_2)].$$

Proof. If n_1 and n_2 are relatively prime, then we have $N = n_1 n_2$ in Theorem 21. Therefore $\sigma(L_0, R_0, n_1 n_2) = [\sigma({}^1L_0, {}^1R_0, n_1), \sigma({}^2L_0, {}^2R_0, n_2)]$, $\forall\ (n_1, n_2) = 1$ □

6. Period of the game modulo an integer

For any integer j with its prime factorization, say $j = \prod_{i=1}^{s} p_i^{k_i}$. We can apply Corollary 22 to decompose the period of game modulo j in Theorem 23.

Theorem 23. *Let* $\begin{cases} L_0 \equiv {}^iL_0 & (\text{mod } p_i^{k_i}) \\ R_0 \equiv {}^iR_0 & (\text{mod } p_i^{k_i}) \end{cases}$. *For any integer j with prime factorization $j = \prod_{i=1}^{s} p_i^{k_i}$,*

$$\sigma(L_0, R_0, j) = [\sigma({}^1L_0, {}^1R_0, p_1^{k_1}), \sigma({}^2L_0, {}^2R_0, p_2^{k_2}), \ldots, \sigma({}^sL_0, {}^sR_0, p_s^{k_s})]$$

Proof. Let $j = p_1^{k_1} p_2^{k_2} \ldots p_s^{k_s} = \prod_{i=1}^{s} p_i^{k_i}$, for some prime numbers p_i.
We are going to prove the proposition by conducting mathematical induction on s.
The proposition is true when $s = 2$ by Corollary 22 as $(p_1^{k_1}, p_2^{k_2}) = 1$.
Assume the proposition is true when $s = h$ for some h,
i.e. $\sigma(L_0, R_0, p_1^{k_1} p_2^{k_2} \ldots p_h^{k_h}) = [\sigma({}^1L_0, {}^1R_0, p_1^{k_1}), \sigma({}^2L_0, {}^2R_0, p_2^{k_2}), \ldots, \sigma({}^hL_0, {}^hR_0, p_h^{k_h})]$
Consider the case when $s = h + 1$,

$$\begin{aligned}
\sigma(L_0, R_0, j) &= \sigma(L_0, R_0, p_1^{k_1} p_2^{k_2} \ldots p_{h+1}^{k_{h+1}}) \\
&= \sigma(L_0, R_0, (p_1^{k_1} p_2^{k_2} \ldots p_h^{k_h}) p_{h+1}^{k_{h+1}}) \\
&= [\sigma(L_0, R_0, p_1^{k_1} p_2^{k_2} \ldots p_h^{k_h}), \sigma({}^{h+1}L_0, {}^{h+1}R_0, p_{h+1}^{k_{h+1}})] \\
&= [[\sigma({}^1L_0, {}^1R_0, p_1^{k_1}), \sigma({}^2L_0, {}^2R_0, p_2^{k_2}), \ldots, \sigma({}^hL_0, {}^hR_0, p_h^{k_h})], \\
&\qquad \sigma({}^{h+1}L_0, {}^{h+1}R_0, p_{h+1}^{k_{h+1}})] \\
&= [\sigma({}^1L_0, {}^1R_0, p_1^{k_1}), \sigma({}^2L_0, {}^2R_0, p_2^{k_2}), \ldots, \sigma({}^hL_0, {}^hR_0, p_h^{k_h}), \sigma({}^{h+1}L_0, {}^{h+1}R_0, p_{h+1}^{k_{h+1}})]
\end{aligned}$$

The proposition is also true for $s = h + 1$.
Therefore, the proposition is true for all positive integer s by the principle of mathematical induction. □

7. Period of the game modulo a power of prime

The last step in this paper is to investigate the period of the game modulo a power of a prime number.

Lemma 24. *Let c be a positive integer and let $\sigma_{p^n} = \sigma(L_0, R_0, p^n)$.*
If $\begin{cases} L_{\sigma_{p^n}} &= L_0 + p^n x_1 \\ R_{\sigma_{p^n}} &= R_0 + p^n x_x \end{cases}$ *for some x_1 and x_2, then there exists two integer x_3 and x_4 both are divisible by p such that* $\begin{cases} L_{c\sigma_{p^n}} &= L_0 + cp^n x_1 + p^{n+1} x_3 \\ R_{c\sigma_{p^n}} &= R_0 + cp^n x_2 + p^{n+1} x_4 \end{cases}$

Proof. Let $\sigma_{p^n} = \sigma(L_0, R_0, p^n)$. In this proof, x_i's are integral constants.
Since $\begin{cases} L_{\sigma_{p^n}} &= F_{2\sigma_{p^n}+1} L_0 + F_{2\sigma_{p^n}} R_0 \equiv L_0 \pmod{p^n} \\ R_{\sigma_{p^n}} &= F_{2\sigma_{p^n}} L_0 + F_{2\sigma_{p^n}-1} R_0 \equiv R_0 \pmod{p^n} \end{cases}$,
we may let $\begin{cases} L_{\sigma_{p^n}} &= L_0 + p^n x_1 \\ R_{\sigma_{p^n}} &= R_0 + p^n x_2 \end{cases}$.
We now conduct mathematical induction on c to prove the proposition.
When $c = 1$, the proposition is true taking $x_3 = x_4 = 0$.

Assume the proposition is true for some $c = k$.
i.e. $\begin{cases} L_{k\sigma_{p^n}} &= F_{2k\sigma_{p^n}+1} L_0 + F_{2k\sigma_{p^n}} R_0 = L_0 + kp^n x_1 + p^{n+1} x_3 \\ R_{k\sigma_{p^n}} &= F_{2k\sigma_{p^n}} L_0 + F_{2k\sigma_{p^n}-1} R_0 = R_0 + kp^n x_2 + p^{n+1} x_4 \end{cases}$
Now consider the case when $c = k + 1$.

$$\begin{aligned}
L_{(k+1)\sigma_{p^n}} &= F_{2(k+1)\sigma_{p^n}+1} L_0 + F_{2(k+1)\sigma_{p^n}} R_0 \\
&= F_{2\sigma_{p^n}+1}(L_{k\sigma_{p^n}}) + F_{2\sigma_{p^n}}(R_{k\sigma_{p^n}}) \\
&= F_{2\sigma_{p^n}+1}(L_0 + kp^n x_1 + p^{n+1} x_3) + F_{2\sigma_{p^n}}(R_0 + kp^n x_2 + p^{n+1} x_4) \\
&= (F_{2\sigma_{p^n}+1} L_0 + F_{2\sigma_{p^n}} R_0) + kp^n (F_{2\sigma_{p^n}+1} x_1 + F_{2\sigma_{p^n}} x_2) \\
&\quad + p^{n+1}(F_{2\sigma_{p^n}+1} x_3 + F_{2\sigma_{p^n}} x_4) \\
&= L_{\sigma_{p^n}} + kp^n (x_1 + p^n x_1') + p^{n+1}(x_3 + p^n x_3') \\
&= (L_0 + p^n x_1) + kp^n (x_1 + p^n x_1') + p^{n+1}(x_3 + p^n x_3') \\
&= L_0 + (k+1)p^n x_1 + p^{n+1}(kp^{n-1} x_1' + x_3 + p^n x_3') \\
L_{(k+1)\sigma_{p^n}} &= L_0 + (k+1)p^n x_1 + p^{n+1} x_3'' \qquad \text{where } x_3'' \text{ is divisible by } p
\end{aligned}$$

Similarly

$$\begin{aligned}
R_{(k+1)\sigma_{p^n}} &= F_{2(k+1)\sigma_{p^n}} L_0 + F_{2(k+1)\sigma_{p^n}-1} R_0 \\
&= F_{2\sigma_{p^n}}(L_{k\sigma_{p^n}}) + F_{2\sigma_{p^n}-1}(R_{k\sigma_{p^n}}) \\
&= F_{2\sigma_{p^n}}(L_0 + kp^n x_1 + p^{n+1} x_3) + F_{2\sigma_{p^n}-1}(R_0 + kp^n x_2 + p^{n+1} x_4) \\
&= (F_{2\sigma_{p^n}} L_0 + F_{2\sigma_{p^n}-1} R_0) + kp^n (F_{2\sigma_{p^n}} x_1 + F_{2\sigma_{p^n}-1} x_2) \\
&\quad + p^{n+1}(F_{2\sigma_{p^n}} x_3 + F_{2\sigma_{p^n}-1} x_4) \\
&= (R_{\sigma_{p^n}}) + kp^n (x_2 + p^n x_2') + p^{n+1}(x_4 + p^n x_4') \\
&= (R_0 + p^n x_2) + kp^n (x_2 + p^n x_2') + p^{n+1}(x_4 + p^n x_4') \\
&= R_0 + (k+1)p^n x_2 + p^{n+1}(kp^{n-1} x_2' + x_4 + p^n x_4')
\end{aligned}$$

$$R_{(k+1)\sigma_{p^n}} = R_0 + (k+1)p^n x_2 + p^{n+1}x_4'' \qquad \text{where } x_4'' \text{ is divisible by } p$$

The proposition is also true for $c = k+1$.
By the principle of mathematical induction, the proposition is true for all positive integer c. □

Lemma 25. *Let $\sigma_{p^k} = \sigma(L_0, R_0, p^k)$ and $\begin{cases} L_{\sigma_{p^n}} = L_0 + p^n x_1 \\ R_{\sigma_{p^n}} = R_0 + p^n x_2 \end{cases}$ for some x_1 and x_2.*

(a) $\sigma_{p^n} \mid \sigma_{p^{n+1}}$ and $\sigma_{p^{n+1}} \mid p\sigma_{p^n}$.
(b) $\sigma_{p^{n+1}} = \begin{cases} \sigma_{p^n} & \text{if } p \mid x_1 \wedge p \mid x_2 \\ p\sigma_{p^n} & \text{if } (p, x_1) = 1 \vee (p, x_2) = 1 \end{cases}$.

Proof. (a) Let $\begin{cases} L_{\sigma_{p^{n+1}}} = L_0 + p^{n+1}x_1 \\ R_{\sigma_{p^{n+1}}} = R_0 + p^{n+1}x_2 \end{cases}$ for some x_1 and x_2.

Then $\begin{cases} L_{\sigma_{p^{n+1}}} \equiv L_0 \pmod{p^n} \\ R_{\sigma_{p^{n+1}}} \equiv R_0 \pmod{p^n} \end{cases}$ and hence $\sigma_{p^n} \mid \sigma_{p^{n+1}}$

By Lemma 24, we have $\begin{cases} L_{c\sigma_{p^n}} = L_0 + cp^n x_1 + p^{n+1}x_3 \\ R_{c\sigma_{p^n}} = R_0 + cp^n x_2 + p^{n+1}x_4 \end{cases}$

Putting $c = p$, we get

$$\begin{cases} L_{p\sigma_{p^n}} = L_0 + p^{n+1}x_1 + p^{n+1}x_3 \equiv L_0 \pmod{p^{n+1}} \\ R_{p\sigma_{p^n}} = R_0 + p^{n+1}x_2 + p^{n+1}x_4 \equiv R_0 \pmod{p^{n+1}} \end{cases}. \qquad (**)$$

Hence $\sigma_{p^{n+1}} \mid p\sigma_{p^n}$.

(b) (i) If $p \mid x_1$ and $p \mid x_2$, then $x_1 = pk_1$ and $x_2 = pk_2$ for some k_1 and k_2.
So $\begin{cases} L_{\sigma_{p^n}} = L_0 + p^{n+1}k_1 \equiv L_0 \pmod{p^{n+1}} \\ R_{\sigma_{p^n}} = R_0 + p^{n+1}k_2 \equiv R_0 \pmod{p^{n+1}} \end{cases}$
Therefore $\sigma_{p^{n+1}} \mid \sigma_{p^n}$. By (a), $\sigma_{p^n} = \sigma_{p^{n+1}}$.

(ii) Assume $(p, x_1) = 1 \vee (p, x_2) = 1$. By Lemma 24,

$$\begin{cases} L_{\sigma_{p^n}} = L_0 + p^n x_1 \\ R_{\sigma_{p^n}} = R_0 + p^n x_2 \end{cases} \Rightarrow \begin{cases} L_{c_n\sigma_{p^n}} = L_0 + c_n p^n x_1 + p^{n+1}x_3 \\ R_{c_n\sigma_{p^n}} = R_0 + c_n p^n x_2 + p^{n+1}x_4 \end{cases}$$

Note that $\begin{cases} L_{c_n\sigma_{p^n}} = L_{\sigma_{p^{n+1}}} \equiv L_0 \pmod{p^{n+1}} \\ R_{c_n\sigma_{p^n}} = R_{\sigma_{p^{n+1}}} \equiv R_0 \pmod{p^{n+1}} \end{cases}$, therefore c_n is a multiple of p. Knowing that we must take the smallest possible value for c_n, we have $c_n = p, \sigma_{p^{n+1}} = p\sigma_{p^n}$.

□

Theorem 26. *If $\sigma_{p^{k+1}} = p\sigma_{p^k}$ for some k, then $\sigma_{p^n} = p^{n-k}\sigma_{p^k}$ for all $n \geq k$.*

Proof. By the equations in (**), we have

$$\begin{cases} L_{p\sigma_{p^n}} = L_0 + p^{n+1}x_1 + p^{n+1}x_3 = L_0 + p^{n+1}(x_1 + x_3) \\ R_{p\sigma_{p^n}} = R_0 + p^{n+1}x_2 + p^{n+1}x_4 = R_0 + p^{n+1}(x_2 + x_4) \end{cases}.$$

If $\sigma_{p^{k+1}} = p\sigma_{p^k}$, by Lemma 25(b), we have $\sigma_{p^k} \neq \sigma_{p^k}$ implying $(p, x_1) = 1 \vee (p, x_2) = 1$.
As x_3 and x_4 are divisible by p by Lemma 24, we have $(p, x_1+x_3) = 1 \vee (p, x_2+x_4) = 1$.

By Lemma 24,

$$\begin{cases} L_{\sigma_{p^{k+1}}} = L_{p\sigma_{p^k}} = L_0 + p^{k+1}(x_1 + x_3) \\ R_{\sigma_{p^{k+1}}} = R_{p\sigma_{p^k}} = R_0 + p^{k+1}(x_2 + x_4) \end{cases}$$
$$\Rightarrow \begin{cases} L_{c_{k+1}\sigma_{p^{k+1}}} = L_0 + c_{k+1}p^{k+1}(x_1 + x_3) + p^{k+2}x_5 \\ R_{c_{k+1}\sigma_{p^{k+1}}} = R_0 + c_{k+1}p^{k+1}(x_2 + x_4) + p^{k+2}x_6 \end{cases}.$$

If $\begin{cases} L_{c_{k+1}\sigma_{p^{k+1}}} = L_{\sigma_{p^{k+2}}} \equiv L_0 \pmod{p^{k+2}} \\ R_{c_{k+1}\sigma_{p^{k+1}}} = R_{\sigma_{p^{k+2}}} \equiv R_0 \pmod{p^{k+2}} \end{cases}$, then c_{k+1} must be a multiple of p.
Knowing that we must take the smallest possible value for c_{k+1}, we have $c_{k+1} = p$ and $\sigma_{p^{k+2}} = p\sigma_{p^{k+1}}$. Inductively, we get $\sigma_{p^n} = p^{n-k}\sigma_{p^k}$. □

By Theorem 23, we have

$$\sigma(L_0, R_0, j) = [\sigma({}^1L_0, {}^1R_0, p_1^{k_1}), \sigma({}^2L_0, {}^2R_0, p_2^{k_2}), \ldots, \sigma({}^sL_0, {}^sR_0, p_s^{k_s})].$$

However this result is for the periodicity of the game only, we now try to use some of the theorems in our previous sections to make Theorem 23 applicable for the period of the *generalized Fibonacci sequence* modulo j in Theorem 27.

Theorem 27. *For any integer j with prime factorization $j = \prod_{i=1}^{s} p_i^{k_i}$, there exists a set of s constants, $\{C_i\}$ for $i = 1, 2, \ldots, s$ such that*

$$\pi(R_0, L_0, j) = \mu[\pi(p_1), \pi(p_2), \ldots, \pi(p_s), \prod_{i=1}^{s} p_i^{k_i - C_i}]$$

where $\mu = 2$ or 1 or $\frac{1}{2}$ or $\frac{1}{4}$.

Proof. Let $j = \prod_{i=1}^{s} p_i^{k_i}$, $\sigma_{p^k} = \sigma(L_0, R_0, p^k)$ and $\pi_N = \pi(R_0, L_0, N)$.
By Lemma 25(b),we know that $\sigma_{p^{n+1}} = \sigma_{p^n}$ or $\sigma_{p^{n+1}} = p\sigma_{p^n}$.
Let C_i be the greatest integer such that $\sigma_{p_i^{C_i}} = \sigma_{p_i^{C_i - 1}}$.
By Theorem 26,

If $\sigma_{p^{k+1}} = p\sigma_{p^k}$ for some k, then $\sigma_{p^n} = p^{n-k}\sigma_{p^k}$ for all $n \geq k$.

Therefore we have $\sigma_{p_i^N} = p^{N-C_i}\sigma_{p_i^{C_i}} = p^{N-C_i}\sigma_{p_i^{C_i-1}} = \ldots = p^{N-C_i}\sigma_{p_i}$
Up to this moment, using Theorem 23, we have

$$\begin{aligned}\sigma_j &= [\sigma_{p_i^{k_1}}, \sigma_{p_2^{k_2}}, \ldots, \sigma_{p_s^{k_s}}] \\ &= [p_1^{k_1-C_1}\sigma_{p_1}, p_2^{k_2-C_2}\sigma_{p_2}, \ldots, p_s^{k_s-C_s}\sigma_{p_s}] \\ \sigma_j &= [\sigma_{p_1}, \sigma_{p_2}, \ldots, \sigma_{p_s}, \prod_{i=1}^{s} p_i^{k_i-C_i}]\end{aligned}$$

By Lemma 6,

$$\sigma_n = \begin{cases} \dfrac{\pi_n}{2} & \text{if } \pi_n \text{ is even} \\ \pi_n & \text{if } \pi_n \text{ is odd} \end{cases}$$

Then

$$\begin{aligned}\sigma_j &= [\sigma_{p_1}, \sigma_{p_2}, \ldots, \sigma_{p_s}, \prod_{i=1}^{s} p_i^{k_i-C_i}] \\ &= \mu_1[\pi_{p_1}, \pi_{p_2}, \ldots, \pi_{p_s}, \prod_{i=1}^{s} p_i^{k_i-C_i}]\end{aligned}$$

where $\mu_1 = 1$ or $\dfrac{1}{2}$.
Now by Theorem 16, for some prime p other than 5,

$$\pi_p = \frac{\pi(p)}{2} \quad \text{or} \quad \pi_p = \pi(p)$$

Therefore $[\pi_{p_1}, \pi_{p_2}, \ldots, \pi_{p_s}] = \mu_2[\pi(p_1), \pi(p_2), \ldots, \pi(p_s)]$ where $\nu_2 = 1$ or $\dfrac{1}{2}$

We then have

$$\sigma_j = \mu_1\mu_2[\pi(p_1), \pi(p_2), \ldots, \pi(p_s), \prod_{i=1}^{s} p_i^{k_i-C_i}]$$

Recall the equation

$$\sigma_n = \begin{cases} \dfrac{\pi_n}{2} & \text{if } \pi_n \text{ is even} \\ \pi_n & \text{if } \pi_n \text{ is odd} \end{cases}$$

We may write

$$\pi_j = \mu_1\mu_2\mu_3[\pi(p_1), \pi(p_2), \ldots, \pi(p_s), \prod_{i=1}^{s} p_i^{k_i-C_i}] \text{ where } u_2 = 1 \text{ or } 2$$

Let $\mu = \mu_1\mu_2\mu_3$, we finally have

$$\pi_j = \mu[\pi(p_1), \pi(p_2), \ldots, \pi(p_s), \prod_{i=1}^{s} p_i^{k_i-C_i}] \text{ where } \mu = 2 \text{ or } 1 \text{ or } \frac{1}{2} \text{ or } \frac{1}{4}$$

$\square$

Note: The value of μ is determined by the choice of the initial values and j, yet we fail to find the relationship in between.

8. Conclusion

The scope of this paper is to study the hand game Chopsticks in the beginning. However, we realized that the *generalized Fibonacci sequence* (GFS) can be formed by the game.

The *Fibonacci sequence* can be generalized in many ways. In this paper, we defined the *generalized Fibonacci sequence* $\{F_n\}$ by the recurrence relation

$$F_n = F_{n-1} + F_{n-1}, \quad \text{for all } n \geq 2$$

with $F_0 = a$ and $F_1 = b$, where a and b are fixed non-negative integers and called the initial values of the sequence.

The game involving one player with both hands showing any two numbers (two initial number L_0 and R_0) tapping on each other alternatively, may form the GFS modulo j. We defined the period of the game $\sigma(L_0, R_0, j)$ as the smallest number of steps by which both hand show their initial numbers again. Then we figured out that the period of the game is either the period of GFS modulo j, $\pi(R_0, L_0, j)$ or half of it, depending on whether $\pi(R_0, L_0, j)$ is even or odd. By this relationship, the periodicity of GFS modulo j can be easily found by the periodicity of the game.

We study the game modulo a prime number p first and successfully find that the period of GFS modulo p, $\pi(a, b, p)$ is either $\pi(p)$ or $\frac{\pi(p)}{2}$ for any prime numbers p other than 5, where $\pi(p)$ is the p-th *Pisano Period* in our **Main Theorem 16**. Then we further decompose the period $\sigma(L_0, R_0, j)$ as the least common multiple of the periods of power of prime factors $\sigma(L_0, R_0, p_i^{k_1})$ in **Theorem 23**. We also disclose the periodicity $\pi(a, b, j)$ of *generalized Fibonacci sequence* modulo p other than 5 and prime powers p^k in **Corollary 18-20, Lemma 7 and Theorem 26**. Finally we use **Theorem 27** to construct a general expression for the period of GFS modulo j in terms of the p_i-th *Pisano period*, where p_i's are the prime factors of j.

Although we know the general form for the period of GFS modulo j in **Theorem 27**, we are still uncertain about the constants C_i's and μ in the expression. Despite this little imperfection, the result is such a great leap in history that the period of GFS modulo j has been reduced to the period of *Fibonacci sequence* modulo a prime p.

Appendix A. Strategies on Different Game Situations

A.1. General Rules of the Game

1. Two players show their both hands with only one finger up at the beginning.
2. One of the players will be attacker.
3. Attacker can choose one of his or her hands as weapon to tap one of the three other hands.
4. After that, the number represented by the weapon remains unchanged.
5. At the same time, the target (the hand the weapon taps) will have to change the number, which is the sum of its original number and the weapon.
6. If the number represented by the target is 5, it will be knocked out and cannot be used by the player again.
7. If the number represented by the target is greater than 5, subtract that number by 5.
8. Players will take turns to be attacker.
9. The player with both hands knocked out will lose the game.

A.2. General notation of the game

1. The $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ will be used to represent the number shown by the 4 hands.
2. The first row is the hands of player A while the second row is the hands of player B.
3. When the first row is in red font, it is player A's turn. When the second row is in red font, it is player B's turn.
4. The words above the arrow represent the strategies used.
5. If both sides have only one hand, no words will be on the arrow as no strategies can be used.
6. If the hand is knocked out, it is represented by "*". It cannot be used again.
7. The player with both hands knocked out loses. The blue font will be used to indicate the winner.
8. The notation of the strategies is $\text{Target}_{\text{weapon}}$, e.g. L_l. Here are the six possible strategies:
   a) L_l: tap opponents' left hand by player's own left hand.
   b) L_r: tap opponents' left hand by player's own right hand.
   c) R_l: tap opponents' right hand by player's own left hand.
   d) R_r: tap opponents' right hand by player's own right hand.
   e) D_l: tap player's own right hand by player's left hand.
   f) D_r: tap player's own left hand by player's right hand.
9. We will also use x to denote any weapon of the player, but known target. For example, we use D_x to denote tapping player's own hand by player's another hand in general.
10. We will use A_l, A_r, B_l and B_r to represent the left hand, right hand of player A, and left hand, right hand, of player B respectively.

11. In general, we assume $x = A_l, y = A_r, z = B_l, a = B_r$.
12. Unless specified, we assume player A starts the game.

A.3. Mathematical Meaning of the strategies

In fact, the strategies described above can be expressed mathematically. It is an addition of matrix. The corresponding addition will be shown above.

Strategy	Original	Addition of matrix	Result
L_l	$\begin{pmatrix} x & y \\ a & b \end{pmatrix}$	$\begin{pmatrix} x & y \\ a & b \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ x & 0 \end{pmatrix}$	$\begin{pmatrix} x & y \\ a+x & b \end{pmatrix}$
L_r	$\begin{pmatrix} x & y \\ a & b \end{pmatrix}$	$\begin{pmatrix} x & y \\ a & b \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ y & 0 \end{pmatrix}$	$\begin{pmatrix} x & y \\ a+y & b \end{pmatrix}$
R_l	$\begin{pmatrix} x & y \\ a & b \end{pmatrix}$	$\begin{pmatrix} x & y \\ a & b \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & x \end{pmatrix}$	$\begin{pmatrix} x & y \\ a & b+x \end{pmatrix}$
R_r	$\begin{pmatrix} x & y \\ a & b \end{pmatrix}$	$\begin{pmatrix} x & y \\ a & b \end{pmatrix} + \begin{pmatrix} 0 & 0 \\ 0 & y \end{pmatrix}$	$\begin{pmatrix} x & y \\ a & b+y \end{pmatrix}$
D_l	$\begin{pmatrix} x & y \\ a & b \end{pmatrix}$	$\begin{pmatrix} x & y \\ a & b \end{pmatrix} + \begin{pmatrix} 0 & x \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} x & x+y \\ a & b \end{pmatrix}$
D_r	$\begin{pmatrix} x & y \\ a & b \end{pmatrix}$	$\begin{pmatrix} x & y \\ a & b \end{pmatrix} + \begin{pmatrix} y & 0 \\ 0 & 0 \end{pmatrix}$	$\begin{pmatrix} x+y & y \\ a & b \end{pmatrix}$

After tapping, it is player B’s turn. The red font moves from the first row to the second row.

A.4. Demonstration of the game

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \xrightarrow{R_l}$$
$$\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix} \xrightarrow{L_r} \begin{pmatrix} 1 & 3 \\ * & 1 \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} 2 & 3 \\ * & 1 \end{pmatrix} \xrightarrow{D_r} \begin{pmatrix} * & 3 \\ * & 1 \end{pmatrix} \rightarrow \begin{pmatrix} * & 4 \\ * & 1 \end{pmatrix} \rightarrow \begin{pmatrix} * & 4 \\ * & * \end{pmatrix}$$

A.5. 1 hand vs 1 hand

Ultimately, we will have to investigate the hand game played by two players with both have 2 hands initially. However, before that, we can investigate the situation of 2 players, with both 1 hand first. For any two number x and y, they can be expressed by one of the four cases:

1. $x + y \equiv 0 \pmod 5$
2. $x + 2y \equiv 0 \pmod 5$

3. $x + 3y \equiv 0 \pmod 5$
4. $x + 4y \equiv 0 \pmod 5$

For instance, When $x = 1$ and $y = 1, 1+4(1) = 5 \equiv 0 \pmod 5$. They can be expressed by case 4. When $x = 1$ and $y = 3, 1+3(3) = 10 \equiv 0 \pmod 5$. They can be expressed by case 3.

In the game, as the number represented by the hands cannot exceed 5, it is generally true to say that $x, y \in \{1, 2, 3, 4\}$ when we discuss our game.

In this paper, we use relations like $x + y \equiv 0 \pmod 5$. The case will be the same when the numbers of the 4 hands is multiplied by k, where $k \in \{2, 3, 4\}$, i.e. $x + y \equiv 0 \pmod 5 \Rightarrow (kx) + (ky) \equiv 0 \pmod 5$, where $(k, 5) = 1$. It is true to say that $\begin{pmatrix} 1 & 3 \\ 2 & 1 \end{pmatrix}$ and $\begin{pmatrix} 2 & 1 \\ 4 & 2 \end{pmatrix}$ are the same.

A.6. Exhaustion

Here, we will exhaust all the cases and show the result. As $4 \times 4 = 16$, there are 16 cases. However, we just need to talk about the cases of $\begin{pmatrix} 1 \\ h \end{pmatrix}$, where $h \in \{1, 2, 3, 4\}$. This can be done as the other 12 cases can be generated by multiplying k, where $k \in \{2, 3, 4\}$.

Here are the results:

$$\begin{pmatrix} 1 \\ 1 \end{pmatrix} \to \begin{pmatrix} 1 \\ 2 \end{pmatrix} \to \begin{pmatrix} 3 \\ 2 \end{pmatrix} \to \begin{pmatrix} 3 \\ * \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 2 \end{pmatrix} \to \begin{pmatrix} 1 \\ 3 \end{pmatrix} \to \begin{pmatrix} 4 \\ 3 \end{pmatrix} \to \begin{pmatrix} 4 \\ 2 \end{pmatrix} \to \begin{pmatrix} 1 \\ 2 \end{pmatrix} \to \dots$$

$$\begin{pmatrix} 1 \\ 3 \end{pmatrix} \to \begin{pmatrix} 1 \\ 4 \end{pmatrix} \to \begin{pmatrix} * \\ 4 \end{pmatrix} \qquad \begin{pmatrix} 1 \\ 4 \end{pmatrix} \to \begin{pmatrix} 1 \\ * \end{pmatrix}$$

A.7. Conclusion and proof

From the process of exhaustion, it is easily found that there are a lot of similarities among the results. In fact, the above 16 cases can be grouped into 4 cases, which are closely related to the relations of x and y as discussed above.
Now we will generate the whole sequence:

$$\begin{pmatrix} x \\ y \end{pmatrix} \to \begin{pmatrix} x \\ x+y \end{pmatrix} \to \begin{pmatrix} 2x+y \\ x+y \end{pmatrix} \to \begin{pmatrix} 2x+y \\ 3x+2y \end{pmatrix} \to \begin{pmatrix} 5x+3y \\ 3x+2y \end{pmatrix} \to \begin{pmatrix} 5x+3y \\ 8x+5y \end{pmatrix} \to \begin{pmatrix} 13x+8y \\ 8x+5y \end{pmatrix} \to \dots$$

It can be easily observed that it is a Fibonacci sequence. Let n be the number of steps.
The red font will be: $F_n x + F_{n-1} y$. The game is over when $F_n x + F_{n-1} y \equiv 0 \pmod 5$. Therefore, player A wins the game if the smallest solution of n in $F_n x + F_{n-1} y \equiv 0$

(mod 5) is an even number.
Player B wins the game if the smallest solution of n in $F_n x + F_{n-1} y \equiv 0 \pmod 5$ is an odd number.
For convenience, we use x to represent A_1 and y for B_1.
Case 1:$x + y \equiv 0 \pmod 5$
As $x + y \equiv 0 \pmod 5 \Rightarrow x \equiv -y \pmod 5$,

$$F_n x + F_{n-1} y \equiv 0 \pmod 5 \Rightarrow -F_n y + F_{n-1} y \equiv 0 \pmod 5$$
$$\Rightarrow y(F_{n-1} - F_n) \equiv 0 \pmod 5 \Rightarrow F_n - F_{n-1} \equiv 0$$

$n = 2(F_2 = F_1 = 1)$
In this case, player A wins the game.

Case 2: $x + 2y \equiv 0 \pmod 5$
As $x + 2y \equiv 0 \pmod 5 \Rightarrow x \equiv -2y \pmod 5$,

$$F_n x + F_{n-1} y \equiv 0 \pmod 5 \Rightarrow -2F_n y + F_{n-1} y \equiv 0 \pmod 5$$
$$\Rightarrow y(-2F_n + F_{n-1}) \equiv 0 \pmod 5 \Rightarrow F_{n-1} \equiv 2F_n \pmod 5$$

There are no possible solutions. In fact, an infinite loop will be generated, and the proof is given below:
$P(n) : F_n - 2F_{n+1} = 5R + h$, where R is an integer and h is a positive integer smaller than 5
When $n = 0$,
$LS = 0 - 2 = -2$, which is not divisible by 5.
$P(0)$ is true.

Assume $P(k)$ is true, i.e. $F_k - 2F_{k+1} = 5R + h$
When $n = k + 1$,
$F_{k+1} - 2F_{k+2} = -F_{k+1} - 2F_k = -F_{k+1} - 2(2F_{k+1} + 5R + h) =$
$-5F_{k+1} - 10R - 2h$, which is not divisible by 5
$P(k + 1)$ is true.
By the principle of M.I., $P(n)$ is true for all positive integer n.
Thus, an infinite loop is formed in case 2.

Case 3: $x + 3y \equiv 0 \pmod 5$
As $x + 2y \equiv 0 \pmod 5 \Rightarrow x \equiv -3y \pmod 5$,

$F_n x + F_{n-1} y \equiv 0 \pmod 5 \Rightarrow -3F_n y + F_{n-1} y \equiv 0 \pmod 5 \Rightarrow y(-3F_n + F_{n-1}) \equiv 0 \pmod 5 \Rightarrow F_{n-1} \equiv 3F_n \pmod 5 \quad n = 3\ (3F_3 = 3 \times 2 = 6 \equiv 1 \pmod 5,\ F_2 = 1)$
In this case, player B wins the game.

Case 4: $x + 4y \equiv 0 \pmod 5$
As $x + 4y \equiv 0 \pmod 5 \Rightarrow x \equiv -4y \pmod 5$,

$F_n x + F_{n-1} y \equiv 0 \pmod 5 \Rightarrow -4F_n y + F_{n-1} y \equiv 0 \pmod 5 \Rightarrow y(-4F_n + F_{n-1}) \equiv 0 \pmod 5 \Rightarrow F_{n-1} \equiv 4F_n \pmod 5 \quad n = 4\ (F_4 = 3, 4F_3 = 8 \equiv 3 \pmod 5)$
In this case, player A wins the game.

To conclude, the outcome of the game totally depends on the relation of x and y. Result varies for different cases.

Lemma Aa): For $x + y \equiv 0 \pmod 5$	player A wins the game
Lemma Ab): For $x + 2y \equiv 0 \pmod 5$	It is a draw
Lemma Ac): For $x + 3y \equiv 0 \pmod 5$	player B wins the game
Lemma Ad): For $x + 4y \equiv 0 \pmod 5$	player A wins the game

From above, we have:

A\ B	1	2	3	4
1	A	Draw	B	A
2	B	A	A	Draw
3	Draw	A	A	B
4	A	B	Draw	A

A.8. 1 hand vs 2 hand

Before going into the hand game played by two players with 2 hands, it is better for us to investigate some simpler situation. Here, we assume that a player has 1 hand knocked out, with the enemy still have both hands. Can he fight back?

When we consider this game, it is very important to know that we assume both players will know the best method that is most beneficial to them. For example, if player A uses his left hand to tap player B's hand, he will immediately wins the game. However, when he uses his right hand to tap player B's hand, he cannot win. In this case, we assume player A will have the wise to choose the best route for him, i.e. to win the game, or to avoid losing the game.

As the number represented by the hands cannot exceed 5, it is true to say that $x, y, z, a \in \{1, 2, 3, 4\}$.

It is noticeable that we consider 2 cases when the game is started by different players. $\begin{pmatrix} 1 & 1 \\ 1 & * \end{pmatrix}$ and $\begin{pmatrix} 1 & 1 \\ 1 & * \end{pmatrix}$ are considered as 2 cases. Therefore, to count roughly, there are altogether $4 \times 4 \times 4 \times 2 = 128$ cases. However, as we say before, the case will be the same when the numbers of the 4 hands is multiplied by k, the number of cases are greatly reduced.

Besides, some cases can be reduced, which will be discussed in the corollaries.
On the other hand, we do not consider the situations that $A_l + B_l \equiv 0 \pmod 5$, or $A_r + B_l \equiv 0 \pmod 5$ initially in **Theorem 34**. For cases like $\begin{pmatrix} 1 & 2 \\ 4 & * \end{pmatrix}$, player A can just win the game by using his left hand to tap player B. For cases like $\begin{pmatrix} 1 & 2 \\ 4 & * \end{pmatrix}$,

player B can simply fight back by tapping player A’s left hand. Therefore, it is actually investigating 1 hand vs 1 hand, i.e. $\begin{pmatrix} 2 \\ 4 \end{pmatrix}$, which has been discussed above.

After reducing, there are 12 cases left. They are based on the relations between A_l, A_r and B_l. These 12 cases will be discussed in **Theorem 34**.

A.9. The overview of the 12 cases

1. $\begin{pmatrix} y & y \\ y & * \end{pmatrix}$
2. $\begin{pmatrix} y & y \\ y & * \end{pmatrix}$
3. $\begin{pmatrix} y & y \\ z & * \end{pmatrix}$, where $B_l + 2A_r \equiv 0 \pmod 5$
4. $\begin{pmatrix} y & y \\ z & * \end{pmatrix}$, where $B_l + 2A_r \equiv 0 \pmod 5$
5. $\begin{pmatrix} y & y \\ z & * \end{pmatrix}$, where $B_l + 3A_r \equiv 0 \pmod 5$
6. $\begin{pmatrix} y & y \\ z & * \end{pmatrix}$, where $B_l + 3A_r \equiv 0 \pmod 5$
7. $\begin{pmatrix} y & x \\ y & * \end{pmatrix}$, where $A_l + 2A_r \equiv 0 \pmod 5$
8. $\begin{pmatrix} y & x \\ y & * \end{pmatrix}$, where $A_l + 2A_r \equiv 0 \pmod 5$
9. $\begin{pmatrix} y & x \\ y & * \end{pmatrix}$, where $A_l + 3A_r \equiv 0 \pmod 5$
10. $\begin{pmatrix} y & x \\ y & * \end{pmatrix}$, where $A_l + 3A_r \equiv 0 \pmod 5$
11. $\begin{pmatrix} y & x \\ z & * \end{pmatrix}$, where $A_l + A_r \equiv 0 \pmod 5$
12. $\begin{pmatrix} y & x \\ z & * \end{pmatrix}$, where $A_l + A_r \equiv 0 \pmod 5$

A.10. Lemma, Corollary, Proposition and Theorem

Before talking about different cases, we will give some simple propositions and corollaries so that it is more convenient for us to show our outcome.

Corollary 28. *Interchangeability of L_l, L_r, R_l, R_r*

The initial value of L_l and L_r can interchange. Similar thing can be applied to R_l, R_r. $\begin{pmatrix} x & y \\ z & a \end{pmatrix}$, $\begin{pmatrix} y & x \\ a & z \end{pmatrix}$, $\begin{pmatrix} y & x \\ z & a \end{pmatrix}$ and $\begin{pmatrix} x & y \\ a & z \end{pmatrix}$ should be considered as 1 case. For example, in $\begin{pmatrix} x & y \\ z & a \end{pmatrix}$ if player A takes L_l, it is exactly the same for him to take R_l in $\begin{pmatrix} x & y \\ a & z \end{pmatrix}$.
When $x \neq y$, they cannot both satisfy one case (the four cases). However, they are interchangeable, therefore we express the hand satisfy the case as A_x and B_x. (This may be useful to explain below)

Proposition 29. *If $A_s + B_l \equiv 0 \pmod 5$, player B must tap A_s to avoid losing the game, where $s = l$ or r.*

As $A_s + B_l \equiv 0 \pmod 5$, player B must tap A_s to avoid losing, or A_s will knock him out.

Now, we assume A_s refers to A_r. Here is the situation of player B if he **DOES NOT** tap player A's right hand:

$$\begin{pmatrix} x & y \\ z & * \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} x+z & y \\ z & * \end{pmatrix} \xrightarrow{L_r} \begin{pmatrix} x+z & y \\ * & * \end{pmatrix}$$

Player A taps player B and turn it to $y+z$, player B's hand is knocked out.

Proposition 30. *Given that $B_l + 3A_s \equiv 0$ (mod 5), if player B taps A_s, player A wins, where $s = l$ or r.*

In **Lemma Ac**, we show that player A will lose if $B_l + 3A_s \equiv 0 \pmod 5$ when the game is started by player A. Now, the game is started by player B. If he taps A_s, he is actually doing the same thing as what player A did in **Lemma Ac**. He will lose the game.

Now, we assume $s = r$, here is the situation of player B if he **TAPS** A_r:

$$\begin{pmatrix} x & y \\ z & * \end{pmatrix} \xrightarrow{R_l} \begin{pmatrix} x & y+z \\ z & * \end{pmatrix} \xrightarrow{R_l} \begin{pmatrix} x & y+z \\ * & * \end{pmatrix}$$

Player A taps B_l and turns it to $z+2y$. As $z+2y$[1] $\rightarrow 3z+y \equiv 0 \pmod 5$, player B's hand is knocked out.

These two Propositions are valid in 1 hand vs 2 hand, and general situation in 1 hand vs 2 hand. Special situation will be specified in the later proof.

In **Lemma 31**, we will assume $A_l = A_r$. It will be simpler to deal with 2 variables in the beginning. Note that this lemma is still valid in the cases of $A_l + B_l \equiv 0 \pmod 5$, but **Theorem 34 and Lemma 33** does not (It will be discussed later).

Lemma 31. *Given $A_l = A_r$ initially, if it is B's turn now, A win.*

Proof. Case 1: $B_l + 2A_l \equiv 0 \pmod 5$

$$\begin{pmatrix} y & y \\ z & * \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} z+y & y \\ z & * \end{pmatrix} \xrightarrow{D_l} \begin{pmatrix} z+y & * \\ z & * \end{pmatrix} \rightarrow \begin{pmatrix} 2z+y & * \\ z & * \end{pmatrix} \rightarrow \begin{pmatrix} z+y & * \\ * & * \end{pmatrix}$$

The situation is tricky. As $B_l + 2A_l \equiv 0 \Rightarrow 3B_l + A_l \equiv 0 \pmod 5$, player A can knock out his own right hand, forcing player B to tap his left hand and lose the game.

Case 2: $B_l + 3A_l \equiv 0 \pmod 5$

$$\begin{pmatrix} y & y \\ z & * \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} y+z & y \\ z & * \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} y+z & y \\ * & * \end{pmatrix}$$

Player B must avoid tapping A_l as $3B_l + A_l \equiv 0 \pmod 5$. (**Proposition 30**) In this case, he must tap one of player A's hands, which are both equal to y. Therefore, player B loses.

Case 3: $5A_l \equiv 0 \pmod 5$ $(A_l = B_l)$

$$\begin{pmatrix} y & y \\ y & * \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} 2y & y \\ y & * \end{pmatrix} \xrightarrow{D_l} \begin{pmatrix} 2y & 3y \\ y & * \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} 3y & 3y \\ y & * \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} 3y & 3y \\ 4y & * \end{pmatrix}$$

[1] $z + 2y \equiv 0 \Rightarrow z \equiv -2y \Rightarrow 3z \equiv -6y \Rightarrow 3z \equiv -y \Rightarrow 3z_y \equiv 0 \pmod 5$

After player A takes D_l?, player B must not tap player A's right hand (**Proposition 30**). Nevertheless, after player A taps player B, player B is actually facing the same situation as **case 3**[2]. He loses the game.
(Extra)Case 4: $B_l + A_l \equiv 0 \pmod 5$ (This case is not included in **Theorem 34**)

$$\begin{pmatrix} y & y \\ z & * \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} * & y \\ z & * \end{pmatrix} \xrightarrow{L_r} \begin{pmatrix} * & y \\ * & * \end{pmatrix}$$

Player B must tap A_s to avoid losing the game. (**Proposition 29**) However, he cannot tap both hands of player A. He can just choose one of them to tap. As player B knocks out one of the hands, the other will knock B_l out. Player B eventually loses the game. □

Corollary 32. *Given $A_l = A_r$ initially, if it is A's turn now, A still wins.*

The logic is extremely simple, when player A taps player B by any hand, it will become one of the 4 cases in the above lemma. Player A still wins the game.

Lemma 33. *Given that $A_l + A_r \equiv 0$ (mod 5) and $A_s + B_l \neq 0$, if it is A's turn, A wins. Where $s = l$ or r.*

Proof. As $A_s + B_l \neq 0 \pmod 5$ there are only 1 possible relationship for y and z: $B_l + 2A_l \equiv 0 \pmod 5$, $B_l + 3A_r \equiv 0 \pmod 5$). Given $A_l + A_r \equiv 0 \pmod 5$, as $A_l + B_l \neq 0 \pmod 5$ and $A_r + B_l \neq 0 \pmod 5$, we can make such conclusion as $A_s + B_l \neq 0$, and $B_l \neq A_s$. (if $B_l = A_l, B_l + A_r \equiv 0 \pmod 5$). We can refer to p.2, which shows that 2 cases out of the 4 cases are not true here.
Very obviously, player A can just knock out his own right hand by his left hand, forcing player B to tap A_l and lose. (**Proposition 30**)
Here is the situation described:

$$\begin{pmatrix} x & y \\ z & * \end{pmatrix} \xrightarrow{D_l} \begin{pmatrix} x & * \\ z & * \end{pmatrix} \to \begin{pmatrix} x+z & * \\ z & * \end{pmatrix} \to \begin{pmatrix} x+z & * \\ * & * \end{pmatrix}$$

□

We have talked about 6 cases. The final 5 cases will be discussed under **Theorem 34.**

Theorem 34. *Given that $A_s + B_l \neq 0$ (mod 5) initially, in any cases, A can win the game, where $s = l$ or r.*

Proof. Here are 5 cases not discussed above:
For $A_l = B_l, B_l + 3A_r \equiv 0 \pmod 5$

$$\begin{pmatrix} z & y \\ z & * \end{pmatrix} \text{ and } \begin{pmatrix} z & y \\ z & * \end{pmatrix}$$

A_r

[2] $3y + 3(4y) = 15y \equiv 0 \pmod 5$, this can be treated as $B_l + 3A_l \equiv 0 \pmod 5$

For $A_l = B_l, B_l + 2A_r \equiv 0 \pmod 5$
$\begin{pmatrix} z & y \\ z & * \end{pmatrix}$ and $\begin{pmatrix} z & y \\ z & * \end{pmatrix}$

For $A_l \neq B_l, A_l + A_r \equiv 0 \pmod 5$
$\begin{pmatrix} x & y \\ z & * \end{pmatrix}$
Note:
$\begin{pmatrix} x & y \\ z & * \end{pmatrix}$ has been discussed in **Lemma 33.**

Case 1: For $A_l = B_l, B_l + 3A_r \equiv 0 \pmod 5$, player B first
$\begin{pmatrix} z & y \\ z & * \end{pmatrix}$
If player A takes R_l, as $B_l + 3A_r \equiv 0 \pmod 5$, he loses. (**Proposition 30**)
If player A takes L_l, as $2B_l + A_r \equiv 0 \pmod 5$, it is A's turn, A wins (**Lemma 33**)

Case 2: For $A_l = B_l, B_l + 3A_r \equiv 0 \pmod 5$, player A first
Assume player A uses D_l

$$\begin{pmatrix} z & y \\ z & * \end{pmatrix} \xrightarrow{D_l} \begin{pmatrix} z & z+y \\ z & * \end{pmatrix}$$

Player B must take R_l to avoid a loss. (**Proposition 29**)
After that, the situation becomes **Lemma Ad** $\begin{pmatrix} z & * \\ z & * \end{pmatrix}$. Thus, player A wins the game.

Case 3: $A_l = B_l, B_l + 2A_r \equiv 0 \pmod 5$ with player B first

$\begin{pmatrix} z & y \\ z & * \end{pmatrix}$ $\qquad\qquad$ $\begin{pmatrix} 1 & 3 \\ 1 & * \end{pmatrix}$
Knowing that $2B_l \equiv A_r$[3] $\pmod 5$, if player B takes L_l, he loses. (**Lemma 31**)

$$\begin{pmatrix} z & y \\ z & * \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} y & y \\ z & * \end{pmatrix}$$

However, as it takes R_l, the situation becomes the same as **case 2** in this Theorem[4]
$\begin{pmatrix} z & y \\ z & * \end{pmatrix} \xrightarrow{R_l} \begin{pmatrix} z & z+y \\ z & * \end{pmatrix}$ (same as **case 2**)

Case 4: $A_l = B_l, B_l + 2A_r \equiv 0 \pmod 5$ with player A first

$$\begin{pmatrix} z & y \\ z & * \end{pmatrix} \xrightarrow{D_l} \begin{pmatrix} z & z+y \\ z & * \end{pmatrix}$$

[3] $z + 2y \equiv 0 \pmod 5 \Rightarrow z \equiv -2y \pmod 5 \Rightarrow 2z \equiv -4y \pmod 5 \Rightarrow 2z \equiv y \pmod 5$, i.e. $2B_l \equiv A_r$.

[4] $z + 2y \equiv 0 \Rightarrow 4z + 8y \equiv 0 \Rightarrow 4z + 3y \equiv 0 \Rightarrow z + 3(z+y) \equiv 0 \pmod 5$, i.e. $B_l + 3A_r \equiv 0 \pmod 5$

Assume player A takes D_l:
If player B takes R_l, he loses ($B_l + 3A_r \equiv 0 \pmod 5$), (**Proposition 29**)
However, if A takes L_l, as $2z + (z + y) \equiv 0 \pmod 5$, i.e. $A_l + A_r = 0 \pmod 5$, he will lose the game. (**Lemma 33**)

Case 5: For $A_l \neq B_l, A_r + A_l \equiv 0 \pmod 5$
$\begin{pmatrix} x & y \\ z & * \end{pmatrix}$
The situation is the same as **Lemma 33**, but it is player B's turn. The relation between A_l, A_r and B_l are $B_l + 2A_r \equiv 0 \pmod 5$, $B_l + 3A_l \equiv 0 \pmod 5$
If player B takes L_l, he loses. (**Proposition 30**).
However, if player B takes R_l, as $A_l + B_l \equiv A_r \pmod 5$, he loses. (**Lemma 31**)
Situation when he **TAPS** A_r:

$\begin{pmatrix} x & y \\ z & * \end{pmatrix} \xrightarrow{R_l} \begin{pmatrix} x & x \\ z & * \end{pmatrix}$, which is the same situation as **Lemma 31.**

Finally, all 12 cases are proved. B can win the game if he is wise enough. □

A.11. 2 hand vs 2 hand

Unless specified, here are the positions of x, y, z, a: $\begin{pmatrix} x & y \\ z & a \end{pmatrix}$

Proposition 35. *For $A_l + A_r + B_x$ (refer to above, see if it is ok)$\equiv 0 \pmod 5$, player A must not tap his own hand. Otherwise, he loses. (Special situation will be specified.)*

Logic: If player A taps his own hand, player B will tap that hand also, causing that hand to disappear.

$\begin{pmatrix} x & y \\ z & a \end{pmatrix} \xrightarrow{D_l} \begin{pmatrix} x+y & y \\ z & a \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} * & y \\ z & a \end{pmatrix}$

Lemma 36. *losing condition*

Here I will suggest a situation the starting side will lose the game. Now, it is player A's turn. For $A_l + 3A_r \equiv 0 \pmod 5$,

$$A_l + 3A_r \equiv 0 \pmod 5 \Rightarrow 2A_l + A_r \equiv 0 \pmod 5 \text{ See footnote 1.}$$

Case 1: If player A takes D_l, he loses one hand. (**Proposition 35**)

$\begin{pmatrix} x & y \\ x & x \end{pmatrix} \xrightarrow{D_l} \begin{pmatrix} x & x+y \\ x & x \end{pmatrix} \xrightarrow{R_l} \begin{pmatrix} x & * \\ x & x \end{pmatrix}$

Case 2: If player A takes D_r, he loses one hand. (**Proposition 35**)

$\begin{pmatrix} x & y \\ x & x \end{pmatrix} \xrightarrow{D_r} \begin{pmatrix} x+y & y \\ x & x \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} * & y \\ x & x \end{pmatrix}$

Case 3: If player A takes L_l, he loses one hand. (**Proposition 30**)

$$\begin{pmatrix} x & y \\ x & x \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} x & y \\ 2x & x \end{pmatrix} \xrightarrow{R_l} \begin{pmatrix} x & * \\ x+y & x \end{pmatrix}$$

Case 4: If player A takes L_r, he loses one hand. (**Proposition 30**)

$$\begin{pmatrix} x & y \\ x & x \end{pmatrix} \xrightarrow{L_r} \begin{pmatrix} x & y \\ x+y & x \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} * & y \\ x+y & x \end{pmatrix}$$

For all four cases player A will lose one hand. He will then lose the entire game (**Theorem 34**).

If it is initial situation for player A, he will lose the game. However, for other initial value, it is still possible for players to avoid facing situation like that. This will be discussed in the **Theorem 40**.

Lemma 37. *$A_l + A_r \equiv 0$ (mod 5), $B_l = B_r$, Player A can still avoid losing by a proper move.*

$$\begin{pmatrix} x & y \\ z & z \end{pmatrix}$$

Player A cannot take D_l or D_r (**Proposition 35**)
For case like $A_x = B_x$, player A can win the game.
For convenience for us to explain, we assume A_x refers to A_l.

$$\begin{pmatrix} x & y \\ x & x \end{pmatrix} \xrightarrow{R_r} \begin{pmatrix} x & y \\ x & * \end{pmatrix} \xrightarrow{R_l} \begin{pmatrix} x & * \\ x & * \end{pmatrix}$$

For case like $A_x \neq B_x$, player A only has one choice.
For convenience for us to explain, we assume A_x refers to A_l.
Note that $A_l + 2B_l \equiv 0$ (mod 5), and $A_r + 3B_l \equiv 0$ (mod 5).
Player A must not take L_r, otherwise, he loses. (**Proposition 30**)
However, he can still take L_l to avoid losing the game.

$$\begin{pmatrix} x & y \\ z & z \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} x & y \\ z+x & z \end{pmatrix}$$

Note that as $2B_l + A_l \equiv 0$ (mod 5), and $A_l \neq B_l$. It is actually a situation of $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$ or $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$. This will be discussed in **Lemma 39**.

Lemma 38. *For four different numbers x, y, z, a, and $A_l + B_r \equiv 0$ (mod 5), $A_r + B_l \equiv 0$ (mod 5) initially, both players fail to win to game*

We are now talking about situation like $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$ and $\begin{pmatrix} 4 & 2 \\ 3 & 1 \end{pmatrix}$

$\begin{pmatrix} x & y \\ z & a \end{pmatrix}$, assume it is player A's turn, he has 2 choices.
The relationship are once again $A_l + 2B_l \equiv 0$ (mod 5) and $A_r + B_r \equiv 0$ (mod 5).

We can conclude this as $x \neq y \neq z \neq a$. Two out of four cases are not true here. (Refer to p.2)

Player A must take either R_l or L_r. Otherwise, he loses. (**Proposition 29**)
Here are the consequence.

1. R_l
$\begin{pmatrix} x & y \\ z & a \end{pmatrix} \xrightarrow{R_l} \begin{pmatrix} x & y \\ z & * \end{pmatrix} \xrightarrow{R_l} \begin{pmatrix} x & * \\ z & * \end{pmatrix}$
As $A_l + 2B_l \equiv 0 \pmod 5$, it is an infinite loop. (**Lemma Ab**)
2. L_r
$\begin{pmatrix} x & y \\ z & a \end{pmatrix} \xrightarrow{L_r} \begin{pmatrix} x & y \\ * & a \end{pmatrix} \xrightarrow{R_l} \begin{pmatrix} * & y \\ * & a \end{pmatrix}$
As $A_r + B_r \equiv 0 \pmod 5$, player A will lose the game. (**Lemma Ac**)

To conclude, as player A wants to avoid losing the game, he will choose R_l, forcing the game into an infinite loop.

Lemma 39. *Both side fails to win the game as it enters* $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ *or* $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.

By **Corollary 32**, $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \times 2 \equiv \begin{pmatrix} 1\times 2 & 4\times 2 \\ 2\times 2 & 3\times 2 \end{pmatrix} \equiv \begin{pmatrix} 2 & 3 \\ 4 & 1 \end{pmatrix} \pmod 5$,
Now, we consider $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$.
Taking D_x will lose the game as $1 + 4 \equiv 0 \pmod 5$
It is also undesirable to take R_l and L_r as both $1 + 3(3)$ and $4 + 3(2) \equiv 0 \pmod 5$.
Player A must not do this to avoid losing the game. (**Proposition 30**)
Therefore, there are only two ways for player A to choose, L_l and R_r.

For L_l:
Case 1a)

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} 1 & 4 \\ 3 & 3 \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} 4 & 4 \\ 2 & 3 \end{pmatrix} \xrightarrow{R_l} \begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \rightarrow \dots$$

In step 1, there is another possible route, which will be discussed in case 2.
In step 2, there is another possible route, which will be discussed in case 1c.
In step 3, there is another possible route, which will be discussed in case 1b.
In step 4, player B has no choice but to take R_l (**Lemma 37**).

Case 1b)

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} 1 & 4 \\ 3 & 3 \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} 4 & 4 \\ 3 & 3 \end{pmatrix} \xrightarrow{D_l} \begin{pmatrix} 4 & 3 \\ 3 & 3 \end{pmatrix} \xrightarrow{R_l} \begin{pmatrix} 4 & 1 \\ 3 & 3 \end{pmatrix} \xrightarrow{L_r} \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

In step 1, there are other possible routes, which will be discussed in case 2.
In step 4, it is undesirable for player B to take L_l, as he will eventually lose the game if player A responds by L_r. If player B takes D_x, he loses. (**Proposition**

30)
In step 5, player A has no choice but to take L_r (**Lemma 37**).
The situation of $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$ is the same as $\begin{pmatrix} 2 & 3 \\ 1 & 4 \end{pmatrix}$. (Discussed above)

Case 1c)

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} 1 & 4 \\ 3 & 3 \end{pmatrix} \xrightarrow{R_l} \begin{pmatrix} 1 & 2 \\ 3 & 3 \end{pmatrix} \xrightarrow{R_l} \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$$

According to **Lemma 38**, the game will be turned into an infinite loop.

Case 2)
For R_r:

$$\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix} \xrightarrow{R_r} \begin{pmatrix} 1 & 4 \\ 2 & 2 \end{pmatrix}$$

However, By **Corollary 32**,

$$\begin{pmatrix} 1 & 4 \\ 2 & 2 \end{pmatrix} \times 4 \equiv \begin{pmatrix} 1\times 4 & 4\times 4 \\ 2\times 4 & 3\times 4 \end{pmatrix} \equiv \begin{pmatrix} 4 & 1 \\ 3 & 3 \end{pmatrix} \pmod 5$$

The situation is exactly the same as **case 1**.

Theorem 40. *Both players cannot win the traditional game starting with* $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$ *if they are both wise enough.*

For $x = y = z = a$. i.e. $\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1 & 1 \\ 1 & 1 \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} 1 & 1 \\ 2 & 1 \end{pmatrix} \xrightarrow{L_r} \begin{pmatrix} 2 & 1 \\ 2 & 1 \end{pmatrix} \xrightarrow{R_r} \begin{pmatrix} 2 & 1 \\ 2 & 2 \end{pmatrix} \xrightarrow{L_l} \begin{pmatrix} 4 & 1 \\ 2 & 2 \end{pmatrix} \xrightarrow{L_r} \begin{pmatrix} 4 & 1 \\ 2 & 3 \end{pmatrix}$$

According to **Lemma 39**, the game turns into a draw.
Note that in step 1, it is undesirable to take D_l, as player B will respond by R_l. The situation will be as same as the one in **Lemma 36**, player A will eventually lose the game.
In step 2, player B must take L_r or R_r (the same) to avoid losing. (**Proposition 30**)
In step 3, as $x + y + z \equiv 0 \pmod 5$, it is undesirable to take D_x (**Proposition 35**) R_l is also undesirable. (**Proposition 30**)
In step 4, R_l is a losing step, player A can just respond by L_l.
In step 5, player A just has one choice. (**Lemma 37**)

After that, we will investigate the situation when the game is starting by other initial values.
Situation of A wins:
For of $A_l + B_x \equiv 0 \pmod 5$, $A_r + B_x \not\equiv 0 \pmod 5$, A wins.

For cases like
$\begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix}$, A wins
$\begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix}$, A wins
For situations above, player A can turn the game into the situation in **Lemma 36**, so player B loses the game.

Situation of a draw:
For cases like
$\begin{pmatrix} 1 & 1 \\ 1 & 3 \end{pmatrix}$, it is a draw
$\begin{pmatrix} 1 & 1 \\ 3 & 3 \end{pmatrix}$, it is a draw
$\begin{pmatrix} 1 & 2 \\ 1 & 1 \end{pmatrix}$, it is a draw
$\begin{pmatrix} 1 & 1 \\ 2 & 2 \end{pmatrix}$, it is a draw
$\begin{pmatrix} 1 & 2 \\ 1 & 2 \end{pmatrix}$, it is a draw
$\begin{pmatrix} 1 & 3 \\ 1 & 3 \end{pmatrix}$, it is a draw
$\begin{pmatrix} 1 & 3 \\ 3 & 3 \end{pmatrix}$, it is a draw

For $x \neq y \neq z \neq a$,
i.e. $\begin{pmatrix} 1 & 4 \\ 2 & 3 \end{pmatrix}$, it is a draw.
i.e. $\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, it is a draw. (**Lemma 38**)

Situation of B wins:
The losing situation in **Lemma 36**, i.e. $\begin{pmatrix} 1 & 3 \\ 1 & 1 \end{pmatrix}$ and $\begin{pmatrix} 1 & 2 \\ 2 & 2 \end{pmatrix}$

To conclude, the result of the game is:

$$\begin{cases} \text{A wins, if } x+z \equiv 0 \pmod 5 \text{ and } y+a \neq 0 \pmod 5\text{, or cases like } \begin{pmatrix} 1 & 1 \\ 2 & 3 \end{pmatrix} \text{ and } \begin{pmatrix} 1 & 1 \\ 1 & 2 \end{pmatrix} \\ \text{B wins, if the situation starts with the losing situation in } \textbf{Lemma 15} \\ \text{Draw, if the game stars with other situation} \end{cases}$$

Appendix B. Repeating Strategies of Traditional Hand Games

In a game of two players A and B each having two hands, we may denote the initial value represented by left hand and right hand of A as L_0^A and R_0^A respectively.

These of B are denoted as L_0^B and R_0^B respectively. Then, the values represented by them after nth round are denoted as L_n^A, R_n^A, L_n^B and R_n^B respectively. Then, the different ways of attack are shown as the following table:

Attackers	'Weapons'	'Targets'	Ways of attack
A	L_n^A	R_n^A, L_n^B or R_n^B	6
	R_n^A	L_n^A, L_n^B or R_n^B	
B	L_n^B	L_n^A, R_n^A or R_n^B	6
	R_n^B	L_n^A, R_n^A or L_n^B	

Thus, the number of combinations of ways of attack in a round is $6 \times 6 = 36$. Without loss of generality, we assume that A attacks first. To analyze the properties of the combinations systematically, we arrange them into different groups.

B.1. Group 1: 2 independent flows of fingers

There are 8 combinations in this group. (The tails and tips of the arrows indicate the 'weapons' and 'targets' respectively.)

Situation	
1	$\begin{array}{ccc} L_n^A & \longrightarrow & R_n^A \\ L_n^B & & R_n^B \end{array}$
2	$\begin{array}{ccc} L_n^A & \longleftarrow & R_n^A \\ L_n^B & \longleftarrow & R_n^B \end{array}$
3	$\begin{array}{ccc} L_n^A & \longrightarrow & R_n^A \\ L_n^B & \longleftarrow & R_n^B \end{array}$
4	$\begin{array}{ccc} L_n^A & \longleftarrow & R_n^A \\ L_n^B & \longrightarrow & R_n^B \end{array}$
5	$\begin{array}{cc} L_n^A & R_n^A \\ \downarrow & \uparrow \\ L_n^B & R_n^B \end{array}$

Situation	
6	$\begin{tikzcd} L_n^A & R_n^A \arrow[d] \\ L_n^B \arrow[u] & R_n^B \end{tikzcd}$
7	$\begin{tikzcd} L_n^A \arrow[dr] & R_n^A \\ L_n^B \arrow[ur] & R_n^B \end{tikzcd}$
8	$\begin{tikzcd} L_n^A & R_n^A \arrow[dl] \\ L_n^B & R_n^B \arrow[ul] \end{tikzcd}$

In combination 1, $R_n^A = nL_0^A + R_0^A$ and $R_n^B = nL_0^B + R_0^B$.
When $L_0^A = R_0^A = 1$, then the period equals the modulo j. The properties of the remaining combinations are similar to those of combination 1.

B.2. Group 2: A pair of mutually attacking hands

There are 4 combinations in this group.

Situation	
9	$\begin{tikzcd} L_n^A \arrow[d, leftrightarrow] & R_n^A \\ L_n^B & R_n^B \end{tikzcd}$
10	$\begin{tikzcd} L_n^A & R_n^A \arrow[d, leftrightarrow] \\ L_n^B & R_n^B \end{tikzcd}$
11	$\begin{tikzcd} L_n^A \arrow[dr, leftrightarrow] & R_n^A \\ L_n^B & R_n^B \end{tikzcd}$
12	$\begin{tikzcd} L_n^A & R_n^A \\ L_n^B \arrow[ur, leftrightarrow] & R_n^B \end{tikzcd}$

Obviously, the properties of the 4 pairs of mutually attacking hands are exactly the same as those of self-generating cycles. For those hands having no effect on the game, they remain the same value throughout the game.

B.3. Group 3: Common target

There are 8 combinations in this group.

Situation	
13	$\begin{array}{ccc} L_n^A & & R_n^A \\ \downarrow & & \\ L_n^B & \longleftarrow & R_n^B \end{array}$
14	$\begin{array}{ccc} L_n^A & & R_n^A \\ & \searrow & \\ L_n^B & \longrightarrow & R_n^B \end{array}$
15	$\begin{array}{ccc} L_n^A & & R_n^A \\ & & \downarrow \\ L_n^B & \longrightarrow & R_n^B \end{array}$
16	$\begin{array}{ccc} L_n^A & & R_n^A \\ & \swarrow & \\ L_n^B & \longleftarrow & R_n^B \end{array}$
17	$\begin{array}{ccc} L_n^A & \longrightarrow & R_n^A \\ & & \uparrow \\ L_n^B & & R_n^B \end{array}$
18	$\begin{array}{ccc} L_n^A & \longrightarrow & R_n^A \\ & \nearrow & \\ L_n^B & & R_n^B \end{array}$

Situation	
19	$\begin{array}{ccc} L_n^A & \leftarrow & R_n^A \\ \uparrow & & \\ L_n^B & & R_n^B \end{array}$
20	$\begin{array}{ccc} L_n^A & \leftarrow & R_n^A \\ & \nwarrow & \\ L_n^B & & R_n^B \end{array}$

In combination 13, $L_n^B = n(L_0^A + R_0^B) + L_0^B$
When $L_n^B = L_0^A = R_0^B = 1$,

$$\text{Period}= \begin{cases} j & \text{if } j \text{ is odd} \\ \dfrac{j}{2} & \text{if } j \text{ is even} \end{cases}$$

B.4. Group 4: A single flow of fingers (type 1)

This group contains 6 combinations.

Situation	
21	$\begin{array}{ccc} L_n^A & & R_n^A \\ \downarrow & & \\ L_n^B & \rightarrow & R_n^B \end{array}$
22	$\begin{array}{ccc} L_n^A & & R_n^A \\ & & \downarrow \\ L_n^B & \leftarrow & R_n^B \end{array}$
23	$\begin{array}{ccc} L_n^A & & R_n^A \\ & \searrow & \\ L_n^B & \leftarrow & R_n^B \end{array}$
24	$\begin{array}{ccc} L_n^A & & R_n^A \\ & \swarrow & \\ L_n^B & \rightarrow & R_n^B \end{array}$

Situation	
25	$\begin{tikzcd} L_n^A \arrow[d] & R_n^A \\ L_n^B \arrow[ur] & R_n^B \end{tikzcd}$
26	$\begin{tikzcd} L_n^A & R_n^A \arrow[dl] \\ L_n^B \arrow[u] & R_n^B \end{tikzcd}$
27	$\begin{tikzcd} L_n^A & R_n^A \arrow[d] \\ L_n^B & R_n^B \arrow[ul] \end{tikzcd}$
28	$\begin{tikzcd} L_n^A \arrow[dr] & R_n^A \\ L_n^B & R_n^B \arrow[u] \end{tikzcd}$

In combination 21,

$L_n^A = L_0^A$

$L_n^B = nL_n^A + L_0^B$

$R_n^B = n(L_0^B + L_0^A) + R_0^B$

B.5. A single flow of fingers (type 2)

Situation	
29	$\begin{tikzcd} L_n^A \arrow[r] & R_n^A \\ L_n^B \arrow[u] & R_n^B \end{tikzcd}$
30	$\begin{tikzcd} L_n^A & R_n^A \arrow[l] \\ L_n^B & R_n^B \arrow[u] \end{tikzcd}$
31	$\begin{tikzcd} L_n^A & R_n^A \arrow[l] \\ L_n^B \arrow[ur] & R_n^B \end{tikzcd}$

Situation	
32	$\begin{tikzcd} L_n^A \arrow[dr] & R_n^A \\ L_n^B \arrow[u] & R_n^B \end{tikzcd}$
33	$\begin{tikzcd} L_n^A & R_n^A \arrow[dl] \\ L_n^B & R_n^B \arrow[u] \end{tikzcd}$
34	$\begin{tikzcd} L_n^A \arrow[r] & R_n^A \\ L_n^B & R_n^B \arrow[ul] \end{tikzcd}$
35	$\begin{tikzcd} L_n^A \arrow[d] & R_n^A \\ L_n^B & R_n^B \arrow[ul] \end{tikzcd}$
36	$\begin{tikzcd} L_n^A & R_n^A \arrow[d] \\ L_n^B \arrow[ur] & R_n^B \end{tikzcd}$

In combination 28,

$L_n^B = L_0^B$

$L_n^A = nL_n^B + L_0^A$

$R_n^A = nL_0^A + (n-1)L_0^B + R_0^A$

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Reviewer's Comments

First of all, I would like to congratulate both the students and their guiding teacher for a nice piece of research.

In this document, I would mainly focus on ways to make the Paper even more reader friendly. Of course, these are only my personal opinions for the authors to assess.

Fibonacci sequence is a well known sequence. Fibonacci and Pisano were one and the same person. I think it helps as a gentle introduction to the topic to start with the first few members of the Fibonacci sequence. With the help of EXCEL spreadsheet, this is easily produced:

$n =$	0	1	2	3	4	5	6	7	8	9
$F_n =$	0	1	1	2	3	5	8	13	21	34

We count from $n = 0$. In the sequence we define $F_0 = 0$, $F_1 = 1$, then every subsequent ones are simply equal to the sum of its two predecessors, ie. $F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$. If we take $F_0 = a$, $F_1 = b$, then we call the resulting sequence a Generalised Fibonacci Sequence. If we take the mod 3 of the Fibonacci sequence, for example, then the sequence above becomes:

$n =$	0	1	2	3	4	5	6	7	8	9
$F_n=$	0	1	1	2	3	5	8	13	21	34
$F_n \pmod 3=$	0	1	1	2	0	2	2	1	0	1

Refer to https://en.wikipedia.org/wiki/Pisano_period

It is observed that the Fibonacci sequence module 3 has a period of 8. This period is called Pisano period, denoted by $\pi(3)$. If it is modulo j, it is denoted by $\pi(j)$. One should note that for the Fibonacci sequence to repeat itself, one requires the initial pair 0, 1 to be repeated, as shown in the yellow highlighted cells. With the exception of $\pi(2) = 3$, the Pisano period $\pi(n)$ is always even.

Another example is when j = 11 with initial values 3, 1. The sequence will look like:

$n =$	0	1	2	3	4	5	6	7	8	9	10	11
$F_n =$	3	1	4	5	9	14	23	37	60	97	157	254
$F_n \pmod{11}=$	3	1	4	5	9	3	1	4	5	9	3	1

Note that the period of this Generalised Fibonacci Sequence modulo 11 is 5.

Proof of Lemma 6:

At this point, it is worth noting it would be less confusing if we write:

right	Left
F_0	F_1
F_2	F_3
F_4	F_5
F_6	F_7

But I will leave the notation as it is. From now on, I will assume that the two-column matrix will always look like:

Left	Right
F_1	F_0
F_3	F_2
F_5	F_4
F_7	F_6

In the first example of the usual Fibonacci Sequence modulo 3, the two-column matrix will simply be:

left	Right
1	0
2	1
2	0
1	2
1	0

The position of 0, 1 exactly align with the initial position of 0,1. So period of the two-column matrix is **half** the original period of the Generalised Fibonacci sequence.

In the second example, with initial values 3, 1, the two-column matrix will become:

left	Right
1	3
5	4
3	9
4	1
9	5
1	3

In this case the period of the Generalised Fibonacci Sequence is 5, an odd number, resulting the first repeated number 3 appearing in the wrong column as the initial 3. Hence we need one more cycle to get the column right. So period of the two-column matrix is **equal to** the original period of the Generalised Fibonacci sequence.

This is Lemma 6. It is easier to understand by showing two examples. □

Lemma 7 is a direct consequence of Lemma 6 and the fact that the Pisano period $\pi(n)$ is always even, except for $\pi(2) = 3$. □

Lemma 8.

$$\begin{cases} L_n \neq L_0 \\ R_n \neq R_0 \end{cases} (\operatorname{mod} j) \Rightarrow \begin{cases} kL_n \neq kL_0 \\ kR_n \neq kR_0 \end{cases} (\operatorname{mod} j) \quad \forall\, 0 < n < \pi(L_0, R_0, j)$$

if $(k, j) = 1$.

May need more explanation. May be along this line:
Let $j = p_1^{r_1} p_2^{r_2} \ldots p_m^{r_m}$. Since $(j, k) = 1$, no p_i divides k. if $(L_n - L_0)k \equiv 0\ (mod\ j)$, then all $p_i^{r_i}$ divides $(L_n - L_0)$, so $(L_n - L_0) \equiv 0\ (mod\ j)$.

Proof of Lemma 9.

Lemma 9 follows from $\sigma(L_0 \times 1, L_0 \times 0, j) = \sigma(1, 0, j), \sigma(R_0 \times 0, R_0 \times 1, j) = \sigma(0, 1, j)$ by Lemma 8, provided $(L_0, j) = 1$, and $(R_0, j) = 1$.

Theorem 10 is just the adding up of the L_0 and the R_0 components in Lemma 9.

$$\begin{array}{lc} & \begin{array}{cc} \text{left} & \text{right} \end{array} \\ \begin{array}{l} \text{0th row} \\ \text{1st row} \\ \text{2nd row} \\ \\ \\ \end{array} & \left[\begin{array}{cc} 0 & R_0 \\ R_0 & R_0 \\ 3R_0 & 2R_0 \\ 8R_0 & 5R_0 \\ \vdots & \vdots \end{array}\right] \end{array}$$

↑

should
specify that
0th row
1st row
2nd row

$$\begin{bmatrix} L_n \\ R_n \end{bmatrix} = \begin{bmatrix} F_{2n+1} & F_{2n} \\ F_{2n} & F_{2n-1} \end{bmatrix} \begin{bmatrix} L_0 \\ R_0 \end{bmatrix}.$$

$$\begin{bmatrix} L_1 \\ R_1 \end{bmatrix} = \begin{bmatrix} F_3 & F_2 \\ F_2 & F_1 \end{bmatrix} \begin{bmatrix} L_0 \\ R_0 \end{bmatrix} = \underbrace{\begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}}_{\equiv T} \begin{bmatrix} L_0 \\ R_0 \end{bmatrix}$$

$$\begin{bmatrix} L_2 \\ R_2 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} L_1 \\ R_1 \end{bmatrix}$$

$L_1 \quad R_1$

$2L_1+R_1 \quad L_1+R_1$

so $$\begin{bmatrix} L_n \\ R_n \end{bmatrix} = T^n \begin{bmatrix} L_0 \\ R_0 \end{bmatrix}$$

It is worth noting that the way Fibonacci sequence is constructed, each step is obtained by multiplying the previous step by the matrix $T = \begin{bmatrix} 2 & 1 \\ 1 & 1 \end{bmatrix}$.

There is an easier way to prove Theorem11.

Theorem 10 says $T_n = \begin{bmatrix} F_{2n+1} & F_{2n} \\ F_{2n} & F_{2n-1} \end{bmatrix}$, because $\begin{bmatrix} L_n \\ R_n \end{bmatrix} = \begin{bmatrix} F_{2n+1} & F_{2n} \\ F_{2n} & F_{2n-1} \end{bmatrix} \begin{bmatrix} L_0 \\ R_0 \end{bmatrix}$ and $\begin{bmatrix} L_n \\ R_n \end{bmatrix} = T^n \begin{bmatrix} L_0 \\ R_0 \end{bmatrix}$.

We just need to show that $F_{2n} = (\frac{1+\sqrt{5}}{2})^{2n} - (\frac{1-\sqrt{5}}{2})^{2n}$.

$F_0 = 0,\ F_1 = 1,\ F_n = F_{n-1} + F_{n-2}$ for $n \geq 2$.
$\boxed{\text{guess } F_n = r^n}$. Then "$F_n = F_{n-1} + F_{n-2}$" implies

$$r^n = r^{n-1} + r^{n-2}$$

$$r^2 = r + 1$$

or $$r^2 - r - 1 = 0$$

$$r = \frac{-(-1) \pm \sqrt{(-1)^2 - 4(1)(-1)}}{2} = \frac{1 \pm \sqrt{5}}{2}.$$

General solution of "$F_n = F_{n-1} + F_{n-2}$" is

$$F_n = A\Big(\frac{1+\sqrt{5}}{2}\Big)^n + B\Big(\frac{1-\sqrt{5}}{2}\Big)^n \quad n \geq 2$$

$F_0 = 0 \Rightarrow A + B = 0,\ B = -A$.

$\therefore F_n = A\Big(\frac{1+\sqrt{5}}{2}\Big)^n - A\Big(\frac{1-\sqrt{5}}{2}\Big)^n$

$$F_1 = 1 \Rightarrow 1 = A\Big(\frac{1+\sqrt{5}}{2}\Big)^1 - A\Big(\frac{1-\sqrt{5}}{2}\Big)^1$$

$$1 = \frac{A}{2}\{1 + \sqrt{5} - (1 - \sqrt{5})\} = A\sqrt{5}$$

$$A = \frac{1}{\sqrt{5}}$$

Hence $F_n = \dfrac{1}{\sqrt{5}}\Big\{\Big(\frac{1+\sqrt{5}}{2}\Big)^n - \Big(\frac{1-\sqrt{5}}{2}\Big)^n\Big\}$.

Theorem 11 is proved.