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CIRCLE PACKING

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Abstract. We want to prove that in $\mathbb{R}^2$, the greatest density of unit circle packing is equal to $\frac{\pi}{2\sqrt{3}}$.
Elementary techniques were mainly used in the following proof(s). Dissection method was used to form a house circumscribing the circle. After all cases were considered, the regular hexagonal house was found with the smallest area. Hence, the circle inside this regular hexagonal house is of the highest density, which is $\frac{\pi}{2\sqrt{3}}$.

1. Introduction

Circle packing on plane is an old problem. Many people have thought this problem when they put coins on table. How dense can the coins be put?

Moreover, it is quite practical in our daily lives, like how the fire extinguishers can be put so that they are easily found in fire accidents but smallest number of them is required, how the soccer players can be placed so that they can defense most efficiently and how the rubbish bin can be placed so as to facilitate citizens and to minimize government expenditure.

The problem was partly solved by Fejes Tóth L for infinite area in 1953. However, his proof was as long as 47 lines with a figure and it was certainly not very loquacious. Therefore, we want to solve the same question in a more basic-technique approach but we try to make our proofs as rigorous as Fejes' method.

[1]This work is done under the supervision of the authors' teacher, Mr. Kwok-Kei Chang.

2. Problem

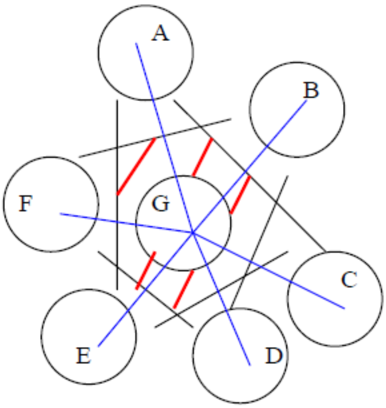
Prove that in $\mathbb{R}^2$, the greatest density of unit circle packing $= \dfrac{\pi}{2\sqrt{3}}$.

We have two different proofs.

2.1. Proof 1

Step 1.

We will use a polygon called convex house to contain each unit circle. Now we introduce how to construct a convex house.



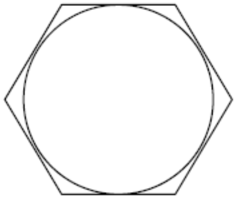
First, we are going to connect the centre of circle A and G, B and G, ..., F and G. Then, we will draw a line which perpendicular to AG and passing through the mid-point of AG. Similarly, a line will be constructed with circle B and G, C and G, ..., F and G. Finally, the convex house will be formed which is the hexagon containing circle G in the above graph.

Step 2.

We will show that the area of all convex houses $\geqslant 2\sqrt{3}$. And this area of house, i.e the smallest area of house can be obtained by hexagonal packing.

Claim: The smallest house = circumscribed regular hexagon containing the unit circle and touching it.

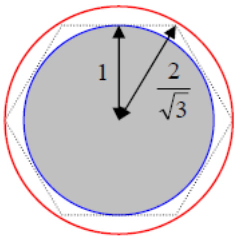
The figure is shown in the following:



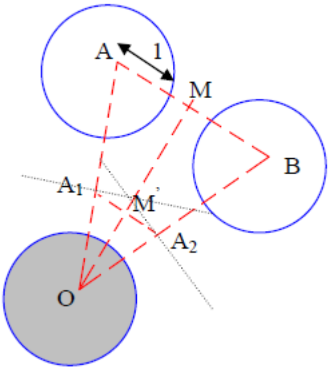
Proof of Step 2.

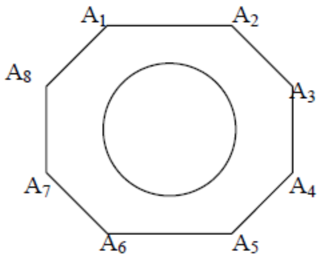
(A) We will construct a convex house which is a circle with radius $\frac{2}{\sqrt{3}}$ with the same center of the unit circle.

The diagram is shown as below:



(B) Now, we will show that at most 7 points A_1, A_2, ..., A_7 can only co-exist within the white region (between the dotted house and the shaded unit circle).





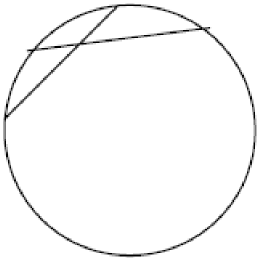
From the graph, OM is perpendicular to A_1A_2 and AB as M is a mid-point of AB. Also, A_1 and A_2 are the mid-point of OA and OB respectively. Clearly, $A_1M' = \frac{1}{2}AM \geqslant \frac{1}{2}$ and $A_2M' = \frac{1}{2}BM' \geqslant \frac{1}{2}$. Therefore, $A_1A_2 \geqslant 1$, as $AB \geqslant 2$.

Similarly, A_2A_3, A_3A_4, ..., $A_nA_{n+1} \geqslant 1$. As the perimeter of the house is $2\pi\left(\frac{2}{\sqrt{3}}\right) = 7.255\ldots$, for 8 points or above, the perimeter of that polygon $\geqslant 8 \geqslant 2\pi\left(\frac{2}{\sqrt{3}}\right)$ = the perimeter of the outer circle(remember the house is convex) which cannot co-exist in the white region. As a result, there are at most 7 points in the white region between the house and the unit circle.

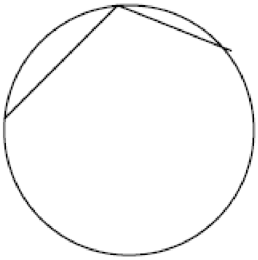
(C) From (B), we know that we have only at most $7A_i$ in the region between outer circle and inner circle. But we only need to consider cases of 6 or $7A_i$ because if there is a 3 side polygon around the unit circle, we can form a 4 side polygon with smaller area around the unit circle, similarly, we can form 5 side, 6 side ... polygons with smaller areas around the unit circle. (Refer to Appendix A).

Now we consider the case of $7A_i$. For this case, the $\perp$ lines the region between outer and inner circles will have 3 situations:

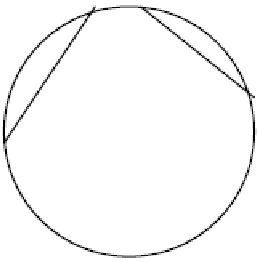
Case I:



Case II:

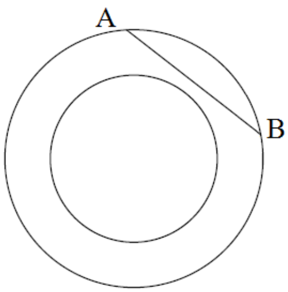


Case III:

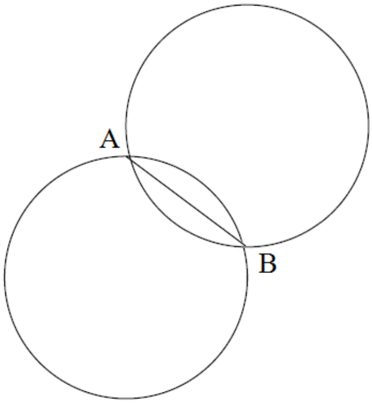


Prove that Case I cannot exist.

First of all, the following figure,



the chord AB can be considered as the common chord between the two identical circles with radii $\frac{2}{\sqrt{3}}$ like as below:



Then, if we have Case I, it means we have three identical circles overlapping together with region CBQ with radii $\frac{2}{\sqrt{3}}$:

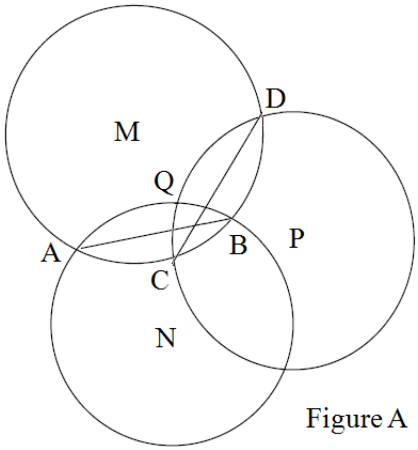
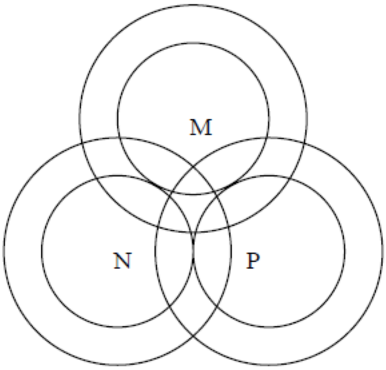


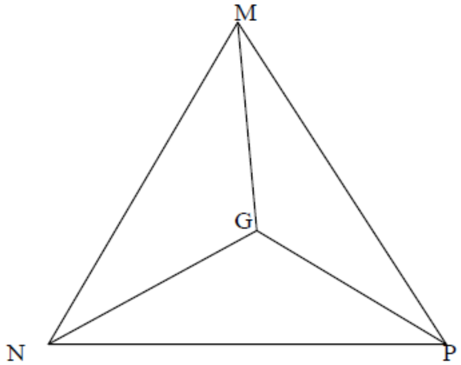
Figure A

AB and CD are the two chords intersecting together with M, N, P be the centres of three circles. However it is impossible. The reason is as follows.

Consider the $\triangle MNP$, the smallest distance for MN, NP, PM is 2 only because these three circles are formed by 3 original circles inside them with radii 1 and these 3 original circles can only touch other in most dense way as below.



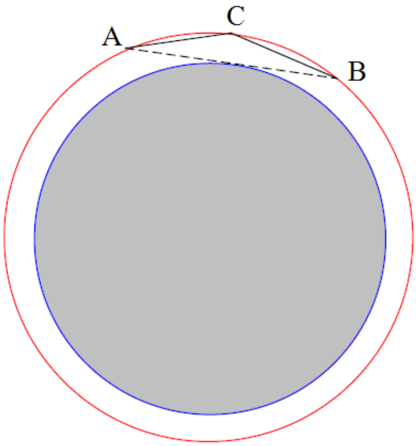
Now, let $\triangle MNP$ be equilateral triangle with side 2 and G be the centroid.



We know that the distance between G and M or N or P is $\dfrac{2}{\sqrt{3}}$ just the radius of the large circle. Clearly, this is for the most dense way to form $\triangle MNP$, for other ways if the circles not packing so dense, $\triangle MNP$ will be larger than this one, and the distance between G and M or N or P will be greater or equal to $\dfrac{2}{\sqrt{3}}$. Then it means the three outer circles with radii $\dfrac{2}{\sqrt{3}}$ can at most intersect together at one point but they cannot intersect together to form a region CBQ like Figure A, i.e. Figure A cannot exist. Then Case I must be rejected.

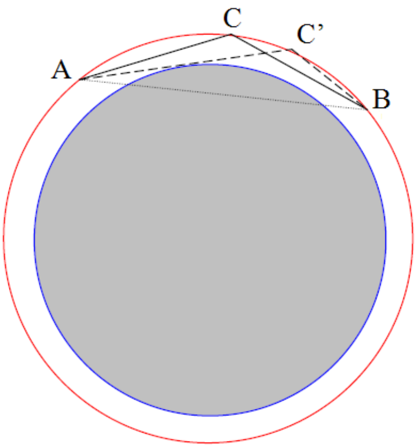
(D) Now consider case II and case III. Here, only seven bisectors can be in the shaded region. They form a heptagon enclosing the inner circle.

Case 1:



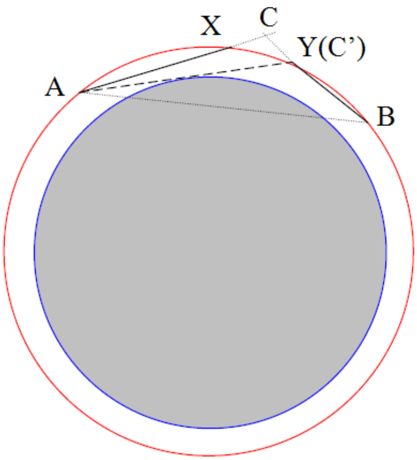
If there exists 3 intersecting points of bisector lines (the wall of the house) A, B, C, which are on the bigger circumference, we can draw a line that is not intersecting with the inner circle, to form a hexagon with one less edge and with $\triangle ABC$'s area minuend. Therefore, the area of heptagon in this case is not the smallest house to contain the circle.

Case 2:



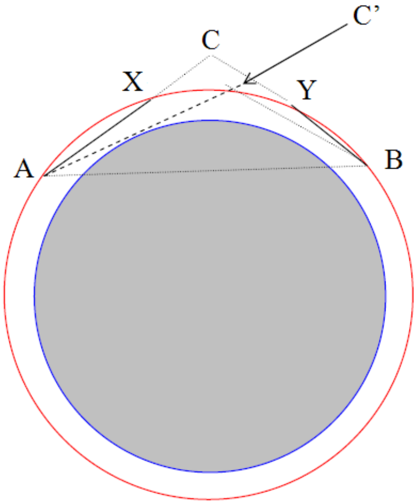
If we cannot draw a line joining AB without intersecting the inner circle, we can draw a tangent of the inner circle from A to C' and connect C' back to B instead. As a matter of fact that C' is of a perpendicular distance to AB shorter than C, the area of $\triangle ABC > \triangle ABC'$. We can then form a new heptagon with ABC' with an area smaller than the original ones (by area of $\triangle ABC-$ area of $\triangle ABC'$). Therefore, the area of heptagon in this case is not the smallest house to contain the circle.

Case 3:



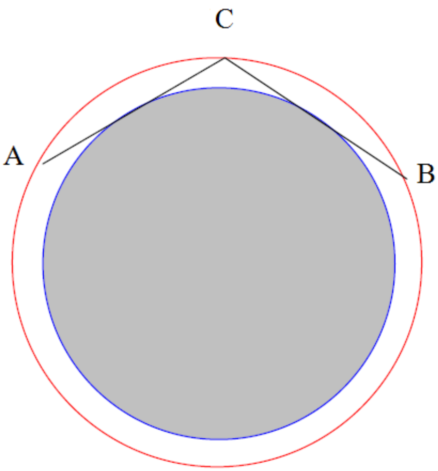
If C is a point that is out of the white region R and we can join AY with out cutting the inner circle to form C', we join AC' that area $\triangle ABC > \triangle ABC'$. A new heptagon is formed with ABC and that will have a smaller area than the original one (by area of $\triangle ABC-$ area of $\triangle ABC'$). Followed by this, the heptagon is of case 1 and 2.

Case 4:

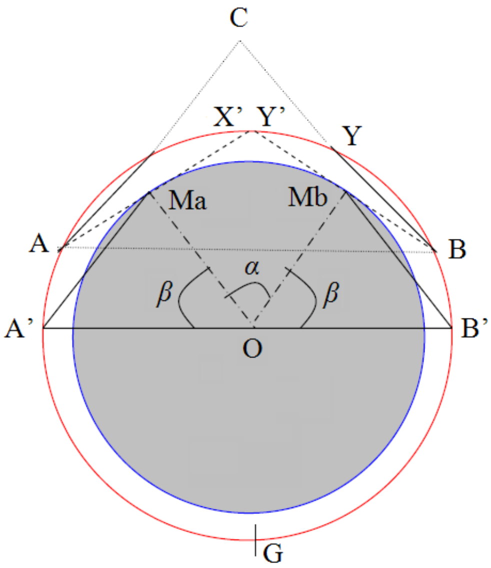


If we can not joint AY without cutting the inner circle, we can join a tangent of inner circle from A and intersect the bigger circle's circumference at C' and C' can also be joint to B without intersecting the inner circle. A new heptagon is formed with ABC' and it is in case 1 and 2.

Case 5:



If A and B can only form two tangents with the inner circle and barely join



together, this case is impossible that the proof is as follows:

1. Ma and Mb are the positions of the bisecting inter-centre distance points (BICD pts) closet to $X'Y'$ from the corresponding circles.
2. $A'Ma = B'Mb = 1 =$ the smallest distance between bisecting BICD pts
3. Consider the case that X' and Y' are just touching each other so that the arc length $A'GB'$ is the greatest and we can put maxium number of BICD pts on the circumference.

Please be noticed that the lines joining the BICD pts are convex as the bisectors has to be joint in convex way.

$$\angle AX'B = \frac{\pi}{3}$$
$$\alpha = \frac{\pi}{3}$$

(it is equivalent to that of circle in a circumscribed hexagon)

$$\beta = \frac{\pi - \cos^{-1}(\frac{A'Ma^2 + OMa^2 - A'O^2}{2A'MaB'Mb})}{2}$$

(by cosine law, base angles of isos. $\triangle$)

$$= \frac{\pi - \cos^{-1}\frac{1}{3}}{2}$$

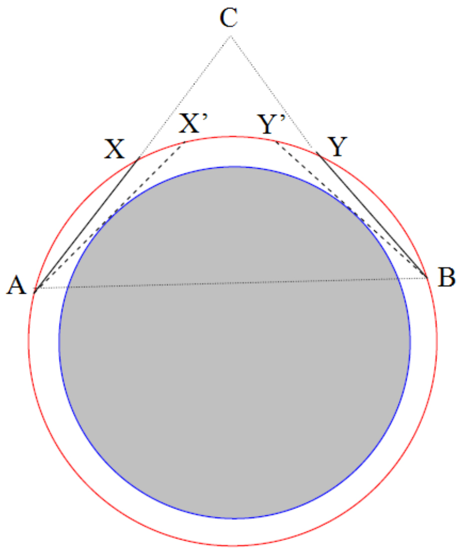
$$\text{The arc Length of } A'GB' = \left(2\pi - \frac{\pi}{3} - \pi + \cos^{-1}\frac{1}{3}\right) \quad \text{(radius)}$$

$$= \left(\frac{2}{3}\pi + \cos^{-1}\frac{1}{3}\right)\left(\frac{2}{\sqrt{3}}\right)$$

$$= 3.839788654 < 4$$

Therefore, the BICD pts cannot have at least 3 points inside the white region. It contradicts to the fact that the inner circle is within heptagon house formed.

The case that the tangents of A and B cannot join within the bigger red circle is also impossible and the proof is similar to the proof in Case 5 mentioned above.



(E) Hence, only 6 points of A_i must be considered. By simple theorem ([1], Chapter 5), we know that the smallest area of hexagon around a circle is a regular hexagon. By simple calculation, the area of a regular hexagon around a unit circle is $2\sqrt{3}$.

Hence, the area of any convex house $\geqslant 2\sqrt{3}$. □

By Step 1 and 2, We can conclude that:

Since the area of any convex house $\geqslant 2\sqrt{3}$,

$$\begin{aligned} & \text{the greatest density of unit circle packing in } \mathbb{R}^2 \text{ plane} \\ = & \text{ area of a unit cirle/area of a regular hexagon house} \\ = & \frac{\pi(1)^2}{2\sqrt{3}} \\ \approx & \ 0.9068997 \end{aligned}$$

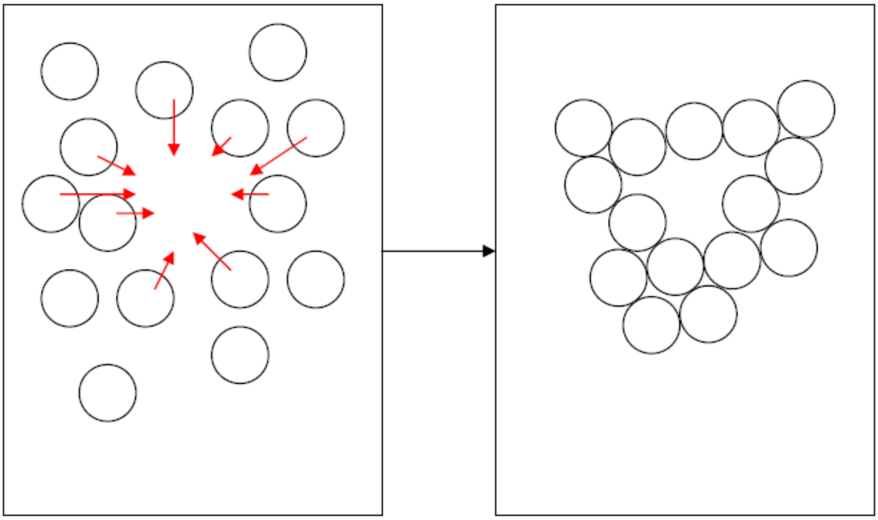
This is the end of Proof 1.

2.2. Proof 2

Here is another proof with a different approach and it is to solve the same problem as the proof above is.

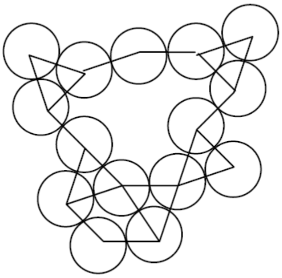
First there exists a group of circles on a planar surface. We want to show that the density is small than $\frac{\pi}{2\sqrt{3}}$, i.e. $= 0.90689968\ldots$ in infinite area. The density of those circles must be smaller than those mapped to be touching with maximum no. of circle is possible (by any means).

Step 1



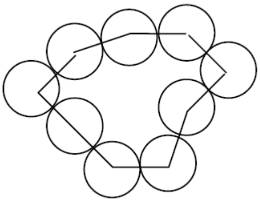
Step 2

We connect those circles' centers by straight lines to the neighboring circles which is touching to each other.



Step 3

We consider a polygon that is formed by the straight lines one by one, like:

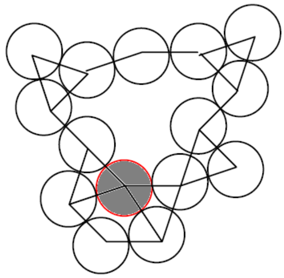


The polygon's area is more than the area occupied by the circles. The number of circles is counted as the sum of the proportion of circle at each angle. Simply,

$$\text{the no. of circles} = \frac{(n-2)\pi}{2\pi} = \frac{n-2}{2}$$

$$\text{(angle sum of polygon } = (n-2)\pi).$$

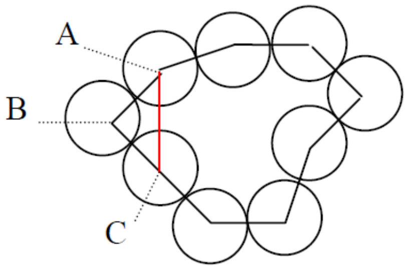
As those circles are in infinite area, each circle must be completely shared by polygons, e.g.



The shaded circle is right shared by 4 polygons. Therefore, there will be no circle miss-counted in this way.

Step 4

First, we consider the smallest angle in the polygon. Of course, the angle must be greater than $\frac{\pi}{3}$. We then connect the centers of the neighouring circles with a straight line to form a new triangle and a new polygon.

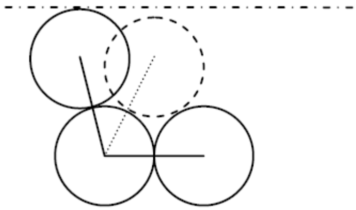


Say ABC is the smallest angle in this polygon. We consider the triangle ABC at the first place.

Case 1: Angle ABC is smaller than $\frac{\pi}{3}$.

As the sum of the interior angles is forever equal to π, changes in the shape of the triangle (with side-lengths greater than $2r$, where r is the radius of the circle) will not affect the area occupied by the circles.

By shifting the circle A to the assigned position (circle with dotted lines), angle ABC is equal to the smallest angle $\frac{\pi}{3}$ and AB, BC, AC are with lengths of the shortest lengths $2r$.



As the area of triangle $= \frac{1}{2}ab\sin\theta$, the original area of triangle must be greater than the new one.

$$\text{The new area } = \frac{1}{2} \times 2r \times 2r \times \sin\frac{\pi}{3} = r^2\sqrt{3}.$$

$$\text{The area the circles occupied } = \frac{\pi \times r^2}{2}.$$

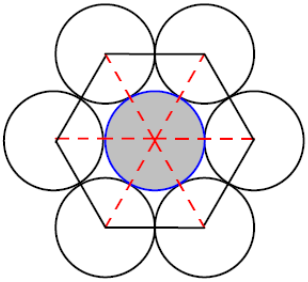
$$\text{The highest density } = \frac{\frac{\pi \times r^2}{2}}{r^2\sqrt{3}} = \frac{\pi}{2\sqrt{3}}.$$

Case 2: The angle ABC is larger than $\frac{2\pi}{3}$.

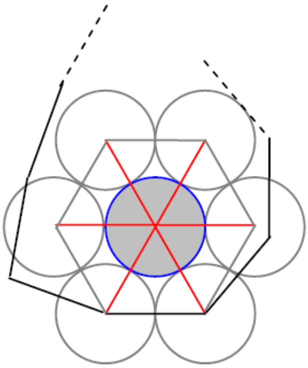
That implies every angle in the polygon is greater than $\frac{2\pi}{3}$. It is only true when the number of sides of the polygon, n, is greater than or equal to 6:

$$(n-2)\pi \geqslant \frac{2\pi}{3}n$$
$$3n - 6 \geqslant 2n$$
$$n \geqslant 6$$

For hexagon ($n = 6$) with angles just equal to $\frac{2\pi}{3}$, it is a regular hexagon which we can put one more circle(the blue one) into without overlapping the circle at the angles. Thus, after putting one more circle into it with touching maximum circle if possible, the density can be increased and we can connect the centers(with dotted lines) again to form new polygons.



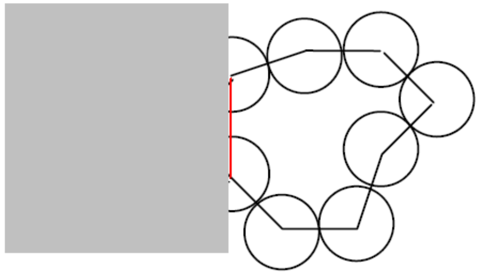
For those polygons with no. of sides greater than 6, since their angle must be larger than $\frac{2\pi}{3}$, the area must be large enough to scribe a hexagon into it. Consider this diagram:



Therefore, one more circle can be scribed into the polygon if its smallest angle is greater than $\frac{2\pi}{3}$ and finally an angle less than $\frac{2\pi}{3}$ is formed.

Step 5

After considering the circle, we can omit the triangle and consider the remaining parts of the polygon.



As the circles connected with the dotted line are not touching to each other, this provides space that the circle can be mapped until touching so that the density can be increased. Those circles will finally touch each other so that one or more new line connecting centre of touching circle can be joint and there must be another polygon formed. Then, we can consider the new polygon from the step 2.

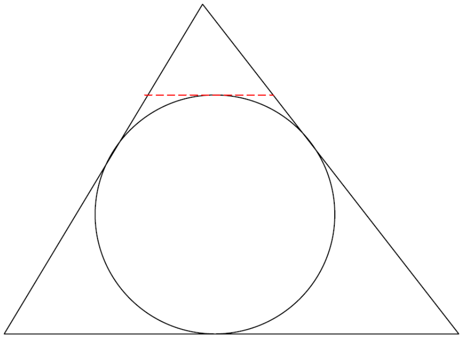
After those steps, say the original polygon is of n sides. Each consideration from step 2 to step 5 will at least omit one side of polygon for next consideration. Therefore, the maximum number of triangle being omitted after "consideration" is $(n-2)$ (as we can only consider 3 circle at the last step). Therefore, the maximum value of density:

$$\begin{aligned}\text{The max. density} &= \frac{\text{the area of total circle inside}}{\text{the area of the polygon}} \\ &= \frac{\frac{n-2}{2}\pi \times r^2}{\text{the total area of } (n-2) \text{ ``smallest triangle''}} \\ &= \frac{\frac{n-2}{2}\pi \times r^2}{\frac{n-2}{2} \times 2r \times 2r \times \sin\frac{\pi}{3}} \\ &= \frac{\pi}{2\sqrt{3}} \\ &= 0.90689968.\end{aligned}$$

This is the end of Proof 2.

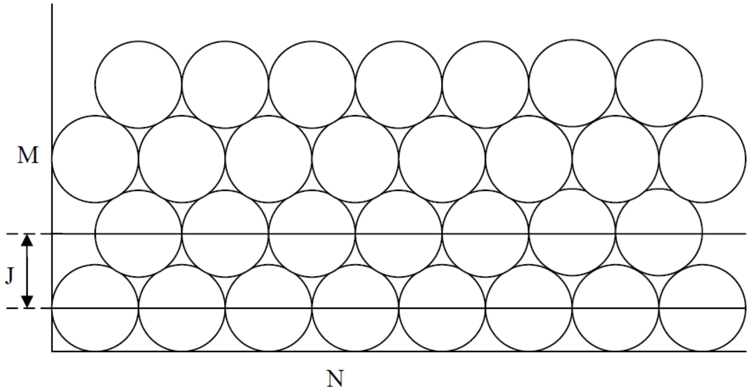
Appendix A.

1. For example, there is a triangle ABC circumscribing the unit circle. We can cut one of the angles as follows, we can form a quadrilateral with smaller area (by cutting the triangle with the dotted line) and i.e. the triangle ABC cannot obtain the smallest area. By similar argument, quadrilateral, pentagon also cannot obtain the smallest area. It means that only hexagon or heptagon can obtain the smallest area circumscribing the unit circle.



2. Further Investigation for Finite Packing:

In $M \times N$ rectangular region, we want to compare which packing for unit circle is denser — rectangular packing or hexagonal packing? We start from arranging circles horizontally, row by row and from bottom to top. We deduce the formula for the number of circles in hexagonal packing. The proof is as follows:



We pack the circles (radius = 1) horizontally first, M, $N > 2$. There are $\left[\frac{N}{2}\right]$ circles in the first row, and $\left[\frac{N-1}{2}\right]$ circles in the second row. $J = 2\sin 60^\circ = \sqrt{3}$. There are $\left[\frac{M-2}{\sqrt{3}}\right] + 1$ columns.

Case 1

$$N - \left[\frac{N}{2}\right] \times 2 \geqslant 1 \quad \text{i.e. } 2r+1 \leqslant N \leqslant 2r+2 \quad r \in \mathbb{Z}$$

$$\text{The no. of balls} = \left[\frac{N}{2}\right]\left(\left[\frac{M-2}{\sqrt{3}}\right] + 1\right).$$

Case 2

$$N - \left[\frac{N}{2}\right] \times 2 \leqslant 1 \quad \text{i.e. } 2r \leqslant N \leqslant 2r+1 \quad r \in \mathbb{Z}$$

$$\begin{aligned}\text{The no. of balls} &= \left[\frac{N}{2}\right]\left(\left[\frac{M-2}{\sqrt{3}}\right] + 1\right) - \left[\frac{\left[\frac{M-2}{\sqrt{3}}\right] + 1}{2}\right] \\ &= \left[\frac{N}{2}\right]\left(\left[\frac{M-2}{\sqrt{3}}\right] + 1\right) - \left[\frac{M-2}{2\sqrt{3}} + \frac{1}{2}\right].\end{aligned}$$

Or simply

$$\text{the no. of balls} = \left[\frac{N}{2}\right]\left(\left[\frac{M-2}{2\sqrt{3}}\right] + 1\right) + \left[\frac{N-1}{2}\right]\left[\frac{M-2}{2\sqrt{3}} + \frac{1}{2}\right].$$

Also, it is easy to find that the formula for the number of circles in rectangular packing is $\left[\frac{M}{2}\right] \times \left[\frac{N}{2}\right]$.

When we compare these two numbers we find that the number for hexagonal packing is greater when $M \geqslant 16$ and $N \geqslant 14$.

We use EXCEL Program to illustrate the result.

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Reviewer's Comments

The reviewer has only comments on the wordings, which have been amended in this paper.