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INVESTIGATION ON BUFFON-LAPLACE NEEDLE PROBLEM

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Abstract. The Buffon-Laplace needle problem is a variation of the well-known Buffon's needle problem, which asks what the probability of a needle, after being dropped to a rectangular grid, intersects, or touches the grid is. In this paper, we aim to solve some generalizations of this problem. We generalized the problem by dropping regular polygons to the grid instead of dropping a needle. We solved this generalization of the problem, by first using the rotational and reflectional symmetry of the regular polygons, then splitting the number of sides of the polygon into 4 cases, then solving each case. We also generalized the problem by dropping arbitrary 2D shapes. We found a general formula and an algorithmic solution to the problem. Apart from generalizations to the problem, we also considered some variations of the problem, like dropping right regular polygon prisms in a 3D space, with the grid being planes in each axis. We used a similar method to solve this problem and provided a formula. We also considered dropping a needle into a n-dimensional space. However, we failed to get a closed form for the formula.

Contents

1. Introduction

1.1. **The Original Problems.**

1.1.1. *The Original Buffon's Neddle Problem.* In the 18th century, Buffon posed the following problem: [Buf33]

"Suppose the floor is paved by parallel stripes of wood of width d. If we drop a needle of length l onto the floor, what is the probability that the needle will lie across a line between two strips?"

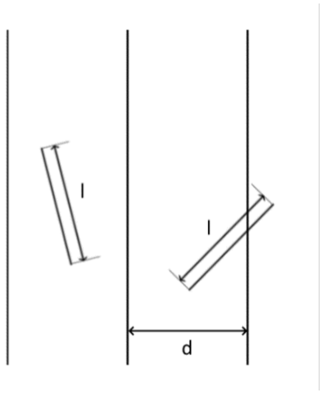


Figure 1. $l \leq d$

This is the original form of the Buffon's Needle problem, and the probability $p(l \leq d)$ is given by

$$p = \frac{2}{\pi}\frac{l}{d}$$

which can be found using probability density functions and integration.

1.1.2. *The Laplace's extension of the Buffon's Needle Problem.* Laplace extended the Buffon's Needle Problem [Lap20] by changing the parallel strips to rectangular tiles with side lengths a and b. The probability p of the needle with length $l(l < a, b)$ lie across one or more lines between two tiles is given by

$$p = \frac{2l(a+b) - l^2}{\pi ab}$$

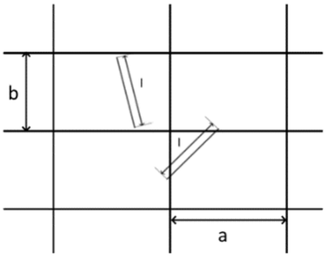


FIGURE 2. $l < a, b$

1.2. **Our Generalizations and Variations on the Laplace-Buffon Needle Problem.** In this paper, we proposed some generalizations of the Laplace-Buffon Needle Problem. We aim to provide a formula or an efficient algorithm for each of those extensions.

1.2.1. *Dropping Regular Polygons.* This is a generalization of the Laplace-Needle problem. Instead of dropping needles to the rectangular grid, we will drop regular polygons with vertices with the distance of one unit away the centre of the polygon. The whole problem is:

Suppose on a 2D plane, there are infinite rectangular tiles with width w and height h (called the grid in this paper), now given a polygon with n sides with each vertex located 1 unit away from the centre of the polygon, if we drop the polygon randomly on the plane, what is the probability p that the polygon intersects with one or more lines between two tiles (called the grid line in this paper)?

We found a formula for p in this problem. We will show our methods in this paper.

1.2.2. *Dropping Arbitrary Polygons.* This is a generalization of the Laplace-Needle problem. Instead of dropping needles to the rectangular grid, we will drop arbitrary polygons. The whole problem is:

Suppose on a 2D plane, there is a rectangular grid with each rectangle of width w and height h, now given a polygon Q with n vertices and the coordinates of the vertices, if we drop Q randomly on the plane, what is the probability p that the polygon intersects with one or more grid line(s)? We have two approaches to this problem. Both approaches consider the fact that:

$$\mathbb{P}(\geq 1 \text{ intersection}) = 1 - \mathbb{P}(0 \text{ intersections})$$

In the first approach, we consider Q rotating about the dropping point (centre) of Q anti-clockwise with angle $\theta(0 \leq \theta < \frac{\pi}{2})$. For each rotation we compute the maximum and minimum values in the x- and y-direction among the vertices, then integrate the joint probability density function of $\theta,\ x,\ y$. We found a formula by this approach. Detailed methods will be shown below.

In the second approach, we consider the rectangular grid line rotating about the origin clockwise with angle $\theta \left(0 \leq \theta < \frac{\pi}{2}\right)$. We found an algorithm by this approach. Detailed explanation of the algorithm will be shown below.

1.2.3. *Dropping Right Regular Prisms in 3D Space.* This is a variation of the Laplace-Needle problem. Instead of dropping needles to the rectangular grid, we will drop right regular prisms to the cuboidal grid in 3D space. The whole problem is:

Suppose in a 3D space, there is a cuboidal grid with each cuboid of length l, width w and height h. Each face of the cuboid forms its corresponding grid plane. Given a right regular prism of height k with $3n$ edges with each vertex located 1 unit away from the nearest centre of the n-sided regular polygon base, if we drop the prism randomly in the space, what is the probability p that the prism intersects with one or more grid plane(s)?

We found a formula for p in this problem. We will show our methods in this paper.

1.2.4. *Generalization to n-dimensions.* This is a generalisation to the Laplace-Needle problem. Instead of dropping needles on a two-dimensional grid, we consider dropping needles on a n-dimensional grid. Unfortunately we could not get a closed form for the formula.

2. Dropping Regular Polygons on Rectangular Grid

Suppose on a 2D plane, there is a rectangular grid with each rectangle of width w and height, now given a polygon with n sides with each vertex located 1 unit away from the centre of the polygon, if we drop the polygon randomly on the plane, what is the probability p that the polygon intersects with one or more grid lines?

We will be using integration on joint probability density functions [PD] as our main method of finding the probability p.

Let's first consider the case $n = 4$, as a demonstration of the method.

2.1. Dropping Squares on Rectangular Grid.

Theorem 2.1. *Given a plane with a rectangular grid with rectangles of width w and height h, and a square of side length l. If the square is dropped randomly on the plane, then the probability p is given by*

$$p = \frac{4l}{\pi h} + \frac{4l}{\pi w} - \frac{l^2(\pi + 2)}{\pi w h}$$

Proof. Let the distance from the geometric centre of the square to the closest horizontal line be $0 \leq x_1 \leq \frac{w}{2}$.

Let the distance from the geometric centre of the square to the closest vertical line be $0 \leq x_2 \leq \frac{h}{2}$.

Let θ be the smallest angle between any side of the square and the horizontal grid lines, so $0 \leq \theta \leq \frac{\pi}{4}$.

x_1, x_2 and θ are independent random variables.

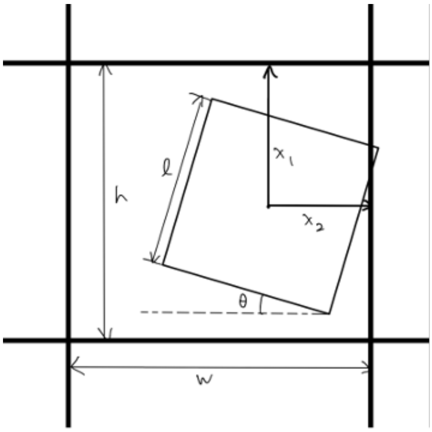


FIGURE 3. Showing the definition of the variables

The uniform probability density function of x_1 and x_2 is:

$$f_{x_1}(i) = \begin{cases} \frac{2}{w} : & 0 \leq i \leq \frac{w}{2} \\ 0 : & \text{otherwise} \end{cases}$$

$$f_{x_2}(j) = \begin{cases} \frac{2}{h} : 0 \leq j \leq \frac{h}{2} \\ 0 : \quad \text{otherwise} \end{cases}$$

The uniform probability density function of θ is:

$$f_{\theta}(k) = \begin{cases} \frac{4}{\pi} : 0 \leq k \leq \frac{\pi}{4} \\ 0 : \text{ otherwise} \end{cases}$$

Referring to Figure 4, the vertical distance from the centre to the point of the

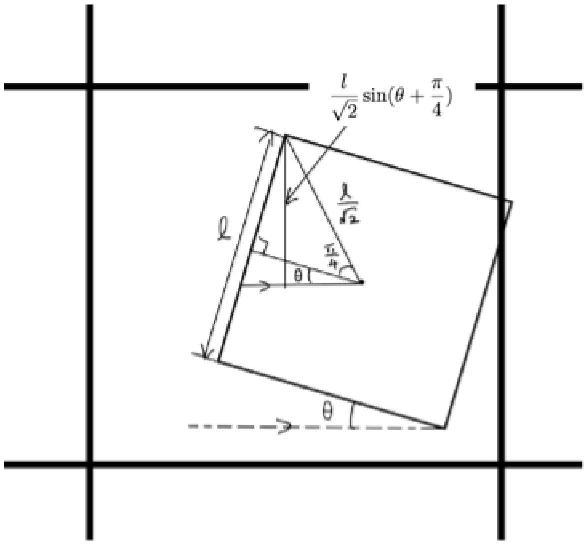


FIGURE 4.

square with the largest y-coordinate and the horizontal distance from the centre to the point of the square with the largest x-coordinate is:

$$\frac{l}{\sqrt{2}} \sin(\theta + \frac{\pi}{4}) = \frac{l}{2}(\sin\theta + \cos\theta)$$

Intersection between the square and the grid lines occurs when:

$$\frac{l}{2}(\sin\theta + \cos\theta) \geq x_1, x_2$$

Since x_1, x_2 and θ are independent random variables, their joint probability density function is:

$$f_{x_1,x_2,\theta}(i,j,k) = \begin{cases} f_{x_1}(i) \cdot f_{x_2}(j) \cdot f_{\theta}(k) = \frac{16}{\pi w h}, & 0 \leq i \leq \frac{w}{2} \wedge 0 \leq j \leq \frac{h}{2} \wedge 0 \leq k \leq \frac{\pi}{4} \\ 0 & \text{otherwise} \end{cases}$$

Therefore,

$$\begin{aligned}
\mathbb{P}(\text{intersection}) &= \mathbb{P}\left(\frac{l}{2}(\sin\theta+\cos\theta) \geq x_1, x_2\right) \\
&= 1-\mathbb{P}\left(\frac{l}{2}(\sin\theta+\cos\theta) < x_1, x_2\right) \\
&= 1-\int_0^{\frac{\pi}{4}}\int_{\frac{l}{2}(\sin\theta+\cos\theta)}^{\frac{h}{2}}\int_{\frac{l}{2}(\sin\theta+\cos\theta)}^{\frac{w}{2}} f_{x_1,x_2,\theta}(i,j,\theta)\,di\,dj\,d\theta \\
&= 1-\int_0^{\frac{\pi}{4}}\int_{\frac{l}{2}(\sin\theta+\cos\theta)}^{\frac{h}{2}}\int_{\frac{l}{2}(\sin\theta+\cos\theta)}^{\frac{w}{2}} \frac{16}{\pi w h}\,di\,dj\,d\theta
\end{aligned}$$

Which is evaluted to be,

$$p = \frac{4l}{\pi h} + \frac{4l}{\pi w} - \frac{l^2(\pi+2)}{\pi w h}$$

□

2.2. **Dropping Arbitrary Shapes on Rectangular Grid.** We will give a formula for computing the probability of a 2D shape intersecting the grid when dropped randomly which serves as a basis for the next section – dropping regular polygons on a rectangular grid.

Given an arbitrary 2D shape Q. Q may be a curve or a polygon. Let V be the list of coordinates of its vertices. If Q is a curve, V may be an infinite list, generated by, perhaps, parametric equations. For example, if Q is a unit circle, $V = \{(i,j) : i,j \in \mathbb{R} \wedge i^2 + j^2 = 1\}$. If Q is a polygon, V is a finite list.

Place Q on a 2D Cartesian plane with the point $(0,0)$ being inside Q. Let the dropping point of Q be the point on Q whose coordinates are (0, 0) on the current Cartesian plane.

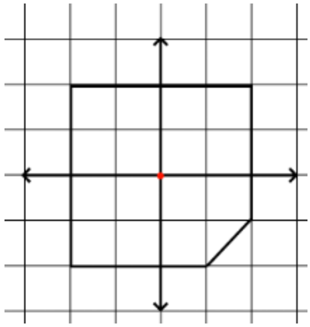


FIGURE 5. V of this figure would be $\{(2,2), (-2,2), (1,-2), (2,-1)\}$

Let the vertical lines on the grid spaced h units apart and horizontal lines w units. When the polygon randomly drops on the grid, without loss of generality, let (x, y) be the coordinates of the dropping point on the grid where $0 \le x < w$ and $0 \le y < h$, and Q rotates around the dropping point $0 \le \theta < \frac{\pi}{2}$ anticlockwise.

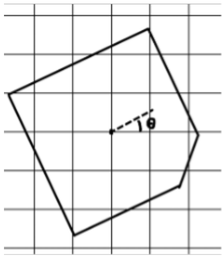


FIGURE 6. Q rotating around dropping point by θ.

Let V_r be the coordinates vertices of Q rotated θ anticlockwise about the dropping point, i.e.,

$$V_r = \{i_x \cos\theta - i_y \sin\theta,\ i_x \sin\theta + i_y \cos\theta\}\ \forall i \in V$$

x, y and θ are independent random variables, meaning they have uniform probability density functions. Their joint probability density functions are hence

$$\begin{cases} \frac{2}{\pi w h} & 0 \le x < w \wedge 0 \le y < h \wedge 0 \le \theta < \frac{\pi}{2} \\ 0 & \text{otherwise} \end{cases} \tag{1}$$

Q **does not** intersect with grid lines when

$$x + \min_i V_{r_{i\,x}} > 0, x + \max_i V_{r_{i\,x}} < w, y + \min_i V_{r_{i\,y}} > 0, y + \max_i V_{r_{i\,y}} < h$$

Hence, we get the following theorem:

Theorem 2.2. *The probability of Q intersecting the grid lines when dropped randomly on a rectangular grid with each rectangle being height h and width w is*

$\mathbb{P}\,(intersect)$

$$= 1 - \int_0^{\frac{\pi}{2}} \int_{-\min_i V_{r_{i\,x}}}^{\max\left(-\min_i V_{r_{i\,x}}, w - \max_i V_{r_{i\,x}}\right)} \int_{-\min_i V_{r_{i\,y}}}^{\max\left(-min_i V_{r_{i\,y}}, h - max_i V_{r_{i\,y}}\right)} \frac{2}{\pi w h} dy\ dx\ d\theta$$

Example 2.3. *When dropping a square of side length l, one possible configuration is*

$$V = \{(l, l), (l, 0), (0, l), (0, 0)\},\ \textit{and}$$

$$V_r = \{(l\cos\theta - l\sin\theta, l\sin\theta + l\cos\theta), (l\cos\theta, l\sin\theta), (-l\sin\theta, l\cos\theta), (0,\ 0)\}$$

Hence,

$$\mathbb{P}\,(intersect) = 1 - \int_0^{\frac{\pi}{2}} \int_{l\sin\theta}^{\max(l\sin\theta,\ w - l\cos\theta)} \int_0^{\max(0,\ h - l(\cos\theta + \sin\theta))} \frac{2}{\pi w h} dy\ dx\ d\theta$$

For $0 \le \theta < \frac{\pi}{2}$,

$$w - l\cos\theta > l\sin\theta \wedge h - l(\cos\theta + \sin\theta) > 0 \textit{ holds only when } l < \frac{w}{\sqrt{2}}, \frac{h}{\sqrt{2}}$$

$$\begin{aligned}\mathbb{P}(intersect) &= 1-\frac{2}{\pi wh}\int_0^{\frac{\pi}{2}}\int_{l\sin\theta}^{w-l\cos\theta}\int_0^{h-l(\cos\theta+\sin\theta)} dy\ dx\ d\theta \\ &= 4l(\frac{1}{\pi h}+\frac{1}{\pi w})-\frac{l^2(\pi+2)}{\pi wh}\end{aligned}$$

This result also agrees with Theorem 2.1.

2.3. **Generalisation to Regular Polygons.** We will reuse the technique of Theorem 2.2. Let Q be a n-sided regular polygon with the distance of each of its vertices to its centre being l.

To simplify calculations, we can scale down the polygon with the distance of each of its vertices to its centre is l, down to 1 by a factor of $\frac{1}{l}$. At the same time, we need to scale down the rectangles in the rectangular grid by the same factor of $\frac{1}{l}$, $d := \frac{d}{l}, w := \frac{w}{l}$. The probability p does not change after the scaling.

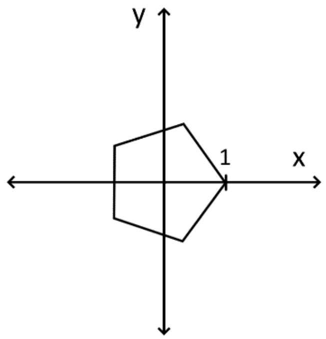


FIGURE 7. Demonstration of setup with Q being a regular pentagon

On the initial Cartesian plane, we situate Q with its geometric centre (centroid) being (0,0) and one of its vertices being (1,0). Refer to Fig. 2.3.1 for a better understanding. Then

$$V=\left\{\left(\cos\frac{2i\pi}{n},\sin\frac{2i\pi}{n}\right)\right\}_{0\le i<n}$$

and

$$V_r=\left\{\left(\cos\left(\frac{2i\pi}{n}+\theta\right),\sin\left(\frac{2i\pi}{n}+\theta\right)\right)\right\}_{0\le i<n}$$

$$where\ \theta\in\left[0,\frac{2\pi}{n}\right).$$

An n-sided polygon has rotational symmetry order n. Therefore, the maximum value for θ is $\frac{2\pi}{n}$, to avoid counting the same rotation twice.

Lemma 2.4. *For a specific rotation of Q, $\theta\in\left[0,\frac{2\pi}{n}\right)$, after Q is flipped vertically, it is the same as Q being rotated by $\frac{2\pi}{n}-\theta$.*

Proof. Let f be a bijective function, $f:\{0,1,2,...,n-1\}\to\{0,1,2,...,n-1\}$.

Let the i-th vertex in Q before flipping vertically become $f(i)$-th vertex after flipping.

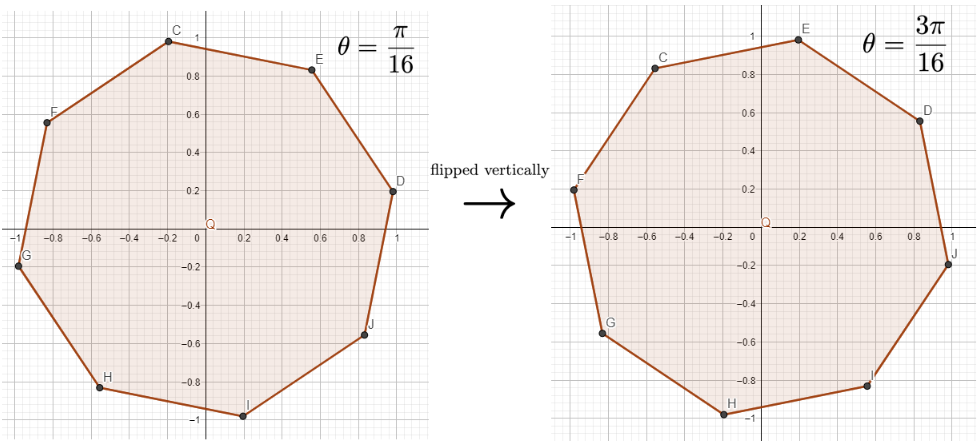
Then Lemma 2.4 states, for $i\in\{0,1,2,...,n-1\}$,

$$\sin\left(\theta + \frac{2i\pi}{n}\right) = -\sin\left(\frac{2\pi}{n} - \theta + \frac{2\pi f(i)}{n}\right)$$

Set $f(i) = n - i - 1$,

$$\begin{aligned} RHS &= \sin\left(\theta + \frac{-2\pi(n-i)}{n}\right) \\ &= \sin\left(\theta + \frac{2i\pi}{n}\right) \\ &= LHS \end{aligned}$$

□



Example 2.5. *Suppose n=8.*

We can observe that Q when $\theta = \frac{\pi}{16}$ *when flipped vertically is exactly Q when* $\theta = \frac{\pi}{4} - \frac{\pi}{16} = \frac{3\pi}{16}$.

Let $\mathbb{P}(\theta \in \{\text{some range}\})$ be the probability for Q ***not*** intersecting with one or more grid lines when θ is in the range given inside the parentheses followed by p.

Lemma 2.6. *For all n,*

$$\mathbb{P}\left(\theta \in \left[0, \frac{2\pi}{n}\right)\right) = 2\mathbb{P}\left(\theta \in \left[0, \frac{\pi}{n}\right)\right)$$

Proof. Using Lemma 2.3, for $\theta \in \left[\frac{\pi}{n}, \frac{2\pi}{n}\right)$, we can flip Q vertically and get a shape with rotation $\theta_{new} = \frac{2\pi}{n} - \theta$, $\theta_{new} \in \left[0, \frac{\pi}{n}\right)$, and has the same probability as the shape is the same as the original shape. Therefore, the probability for Q intersecting with one or more grid lines when $\theta \in \left[0, \frac{\pi}{n}\right)$ is equal to the probability for Q intersecting with one or more grid lines when $\theta \in \left[\frac{\pi}{n}, \frac{2\pi}{n}\right)$, or equivalently,

$$\mathbb{P}\left(\theta \in \left[0, \frac{2\pi}{n}\right)\right) = 2\mathbb{P}\left(\theta \in \left[0, \frac{\pi}{n}\right)\right).$$

□

Lemma 2.7. *If n is odd, for a specific rotation of Q, $\theta \in \left[0, \frac{\pi}{n}\right)$, after Q is flipped horizontally, it is the same as Q being rotated by $\frac{\pi}{n} - \theta$.*

Proof. Let $n = 2m + 1, m \in \mathbb{N}$.
Let g be a bijective function, $g : \{0, 1, 2, ..., n-1\} \rightarrow \{0, 1, 2, ..., n-1\}$.
Let the i-th vertex in Q before flipping horizontally become $g(i)$-th vertex after flipping.
Then Lemma 2.7 states,
For $i \in \{0, 1, 2, ..., n-1\}$,

$$\cos\left(\theta + \frac{2\pi i}{n}\right) = -\cos\left(\frac{\pi}{n} - \theta + \frac{2\pi g(i)}{n}\right)$$

Set $g(i) = \begin{cases} m - i & , 0 \leq i \leq m \\ n + m - i & , \text{otherwise} \end{cases}$,
If $0 \leq i \leq m$,
Substitute $g(i) = m - i$,

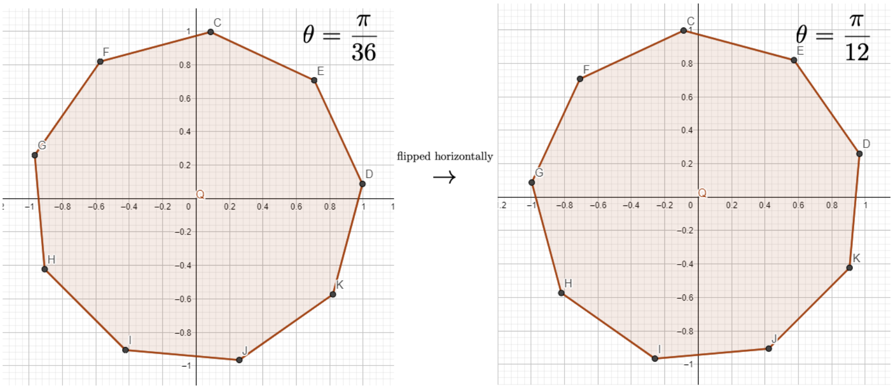
$$\begin{aligned} RHS &= -\cos\left(\frac{\pi}{n} - \theta + \frac{2\pi(m-i)}{n}\right) \\ &= \cos\left(\pi + \theta - \frac{\pi(2m+1)}{n} + \frac{2\pi i}{n}\right) \\ &= \cos\left(\theta + \frac{2\pi i}{n}\right) \\ &= LHS \end{aligned}$$

Otherwise, or if $m + 1 \leq i \leq n - 1$,
Substitute $g(i) = n + m - i$,

$$\begin{aligned} RHS &= -\cos\left(\frac{\pi}{n} - \theta + \frac{2\pi(n+m-i)}{n}\right) \\ &= \cos\left(\pi + \theta - \frac{\pi(2m+1)}{n} + \frac{2\pi i}{n} - \frac{2\pi n}{n}\right) \\ &= \cos\left(\theta + \frac{2\pi i}{n}\right) \\ &= LHS \end{aligned}$$

□

Example 2.8. *Suppose $n = 9$,*
Referring to below figure, we can observe that Q when $\theta = \frac{\pi}{36}$ when flipped horizontally is exactly Q when $\theta = \frac{\pi}{9} - \frac{\pi}{36} = \frac{\pi}{12}$.



Lemma 2.9. *If n is odd, then*

$$\mathbb{P}\left(\theta \in \left[0, \frac{2\pi}{n}\right)\right) = 4\mathbb{P}\left(\theta \in \left[0, \frac{\pi}{2n}\right)\right)$$

Proof. Using Lemma 2.7, for $\theta \in \left[\frac{\pi}{2n}, \frac{\pi}{n}\right)$, we can flip Q horizontally and get a shape with rotation $\theta_{new} = \frac{\pi}{n} - \theta, \theta_{new} \in \left[0, \frac{\pi}{2n}\right)$, and has the same probability as the original shape. Therefore, if n is odd, the probability for Q intersecting with one or more grid lines when $\theta \in \left[0, \frac{\pi}{2n}\right)$ is equal to the probability for Q intersecting with one or more grid lines when $\theta \in \left[\frac{\pi}{2n}, \frac{\pi}{n}\right)$, or equivalently,

$$\mathbb{P}\left(\theta \in \left[0, \frac{\pi}{n}\right)\right) = 2\mathbb{P}\left(\theta \in \left[0, \frac{\pi}{2n}\right)\right)$$

Using Lemma 2.6,

$$\mathbb{P}\left(\theta \in \left[0, \frac{2\pi}{n}\right)\right) = 4\mathbb{P}\left(\theta \in \left[0, \frac{\pi}{2n}\right)\right)$$

$\square$

Now the only thing left is to determine which vertex contributes to the minimum and maximum x and y when rotated, i.e., determine $\min_i V_{r_i x}$, $\max_i V_{r_i x}$, $\min_i V_{r_i y}$, $\max_i V_{r_i y}$.

Lemma 2.10. *For the ranges $\left(\cos\frac{\pi}{n}, 1\right]$ and $\left[-1, -\frac{\cos\pi}{n}\right)$, there is one or zero i such that $V_{r_i x}$ lies inside that range and there is one or zero j such that $V_{r_j y}$ lies inside that range.*

Proof. We prove the statement there is one or zero i such that $V_{r_i x} \in \left(\cos\frac{\pi}{n}, 1\right]$ by contradiction. Suppose $\cos\frac{\pi}{n} < V_{r_i x}, V_{r_j x} \leq 1$ where $0 \leq i, j < n \in \mathbb{Z}, i \neq j$. This means

$$\cos\frac{\pi}{n} < \cos\left(\frac{2i\pi}{n} + \theta\right), \cos\left(\frac{2j\pi}{n} + \theta\right) \leq 1$$

Proof of $\cos\left(\frac{2i\pi}{n} + \theta\right), \cos\left(\frac{2j\pi}{n} + \theta\right) \leq 1$ is trivial as the range of the cosine function is $[-1, 1]$. We now focus on $\cos\frac{\pi}{n} < \cos\left(\frac{2i\pi}{n} + \theta\right), \cos\left(\frac{2j\pi}{n} + \theta\right)$.

When $0 \leq \theta < \frac{\pi}{n}, i = j = 0$ is the only solution - contradiction.
When $\theta = \frac{\pi}{n}, \cos\left(\frac{2i\pi}{n}\right)$ is at most $\cos\frac{\pi}{n}$ when $i = 0$.
When $0 < \frac{\pi}{n} < \frac{3\pi}{n}, i = j = 1$ is the only solution - contradiction.
When $\theta = \frac{3\pi}{n}, \cos\left(\frac{2i\pi}{n} + \theta\right)$ is at most $\cos\frac{\pi}{n}$ when $i = 1$.
Similarly, one can prove for the other ranges.
Hence, $V_{r_j\,x} = \cos\frac{\pi}{n}$ when $\theta = \frac{2k\pi}{n} + \frac{\pi}{n}, k \in \mathbb{Z}$ and there is exactly one $V_{r_j\,x} \in \left(\cos\frac{\pi}{n}, 1\right]$ otherwise.
Proof of the other range and for $V_{r_y\,i}$ are similar.

□

We split into cases according to the value of n modulo 4.

Case 1: $n \equiv 0\,(mod\,4)$

This condition suggests that initially, when $\theta = 0, \max_i V_{r_i\,x}$, $\max_i V_{r_i\,y}$, $\min_i V_{r_i\,x}$, $\min_i V_{r_i\,y}$ are $V_{r_0\,x}$, $V_{r_{\frac{n}{4}}\,y}$, $V_{r_{\frac{n}{2}}\,x}$, $V_{r_{\frac{3n}{4}}\,y} = \cos\theta, \cos\theta, -\cos\theta, -\cos\theta$ respectively. Notice that, for $\theta \in \left[0, \frac{\pi}{n}\right), \cos\theta > \cos\frac{\pi}{n}$. According to Lemma 2.10, $\theta = 0$, $\max_i V_{r_i\,x}$, $\max_i V_{r_i\,y}$, $\min_i V_{r_i\,x}$, $\min_i V_{r_i\,y}$ are $V_{r_0\,x}, V_{r_{\frac{n}{4}}\,y}, V_{r_{\frac{n}{2}}\,x}, V_{r_{\frac{3n}{4}}\,y}$ for all $\theta \in \left(0, \frac{\pi}{n}\right]$ since there are no greater values for the maximums and no lower values for the minimums. Therefore,

$$p = 1 - \int_0^{\frac{2\pi}{n}} \int_{-(-\cos\theta)}^{\max(-(-\cos\theta), w-\cos\theta)} \int_{-(-\cos\theta)}^{\max(-(-\cos\theta), h-\cos\theta)} \frac{n}{2\pi wh}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\theta$$

By Lemma 2.6,

$$\begin{aligned} &= 1 - 2\int_0^{\frac{\pi}{n}} \int_{-(-\cos\theta)}^{\max(-(-\cos\theta), w-\cos\theta)} \int_{-(-\cos\theta)}^{\max(-(-\cos\theta), h-\cos\theta)} \frac{n}{2\pi wh}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\theta \\ &= 1 - \int_0^{\frac{\pi}{n}} \int_{\cos\theta}^{\max(\cos\theta, w-\cos\theta)} \int_{\cos\theta}^{\max(\cos\theta, h-\cos\theta)} \frac{n}{\pi wh}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\theta \end{aligned}$$

If $w, h - \cos\theta < \cos\theta$, then $\frac{w,h}{2} \geq \cos\theta$. Which is only possible when $w, h \leq 2$.
If $w, h \leq 2$,

$$= 1 - \int_{\min\left(\frac{\pi}{n}, \max\left(\arccos\frac{w}{2}, \arccos\frac{h}{2}\right)\right)}^{\frac{\pi}{n}} \int_{\cos\theta}^{w-\cos\theta} \int_{\cos\theta}^{h-\cos\theta} \frac{n}{\pi wh}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\theta$$

When $\theta \leq \max\left(\arccos\frac{w}{2}, \arccos\frac{h}{2}\right), \cos\theta < w - \cos\theta, h - \cos\theta$, so it must intersect with the grid, thus when $\theta \leq \max\left(\arccos\frac{w}{2}, \arccos\frac{h}{2}\right)$, the integral is 0.

$$= 1 - \frac{n}{\pi wh}\left[wh\theta - 2h\sin\theta - 2h\sin\theta + \sin 2\theta + 2\theta\right]_{\min\left(\frac{\pi}{n}, \max\left(\arccos\frac{w}{2}, \arccos\frac{h}{2}\right)\right)}^{\frac{\pi}{n}}$$

If $w, h > 2$,

$$= \frac{2n(h+w)\sin\frac{\pi}{n} - n\sin\frac{2\pi}{n} - 2\pi}{\pi wh}.$$

(By Lemma 2.6 and 2.9)

Example 2.11. *What is the probability p of a polygon with 8 sides, and the distance of each of its vertex to its centre is l , touches one or more grid lines when dropped to a rectangular grid with each rectangle of width w and height h? (w, h > 2l)*

Given $n = 8$. It falls into Case 1 as $n \equiv 0\,(mod\,4)$.

$$w := \frac{w}{l}, h := \frac{h}{l}$$

Using the formula,

$$\begin{aligned}
p &= \frac{8l^2}{\pi wh}\left(\frac{2}{l}(w+h)\sin\pi 8 + \sin\frac{2\pi}{8} - \sin\frac{2\pi}{8} - \frac{2\pi}{8}\right) \\
&= \frac{16l}{\pi}\left(\frac{1}{w}+\frac{1}{h}\right)\sin\frac{\pi}{8} - \frac{l^2}{\pi wh}\left(2\pi + 4\sqrt{2}\right)
\end{aligned}$$

Case 2: $n \equiv 1\,(mod\,4)$

As with Case 1, we could find the vertices with $\max_i V_{r_i x}$, $\max_i V_{r_i y}$, $\min_i V_{r_i x}$, $\min_i V_{r_i y}$ initially. However, note that there is no vertex whose x-coordinate lies inside the range $\left[-1, -\cos\frac{\pi}{n}\right)$, meaning there are two vertices whose x-coordinates are equal to $-\cos\frac{\pi}{n} - V_{\frac{n-1}{2}}$ and $V_{\frac{n+1}{2}}$. But it suffices to take the one with smaller index $-V_{r\frac{n-1}{2}}$ as $\min_i V_{r_i x}$ since $V_{r\frac{n-1}{2}} = \cos\left(\theta + \frac{2\pi(n-1)}{2n}\right) \in \left[-1, -\cos\frac{\pi}{n}\right) \forall\theta \in \left(0, \frac{\pi}{n}\right]$. Therefore, we have $\max_i V_{r_i x}, \max_i V_{r_i y}, \min_i V_{r_i x}, \min_i V_{r_i y} = V_{r_0 x}, V_{r_{\frac{n-1}{4}} y}, V_{r_{\frac{n-1}{2}} x}, V_{r_{\frac{3n+1}{4}}} = \cos\theta, \sin\left(\theta + \frac{2\pi(n-1)}{4n}\right), \cos\left(\theta + \frac{2\pi(n-1)}{2n}\right), \sin\left(\theta + \frac{2\pi(3n+1)}{4n}\right)$. Carefully observing, for all $V_{r_0 x}, V_{r_{\frac{n-1}{2}} x} \in \left(\cos\frac{\pi}{n}, 1\right]$ and $V_{r_{\frac{n-1}{2}} x}, V_{r_{\frac{3n+1}{2}}} \in \left[-1, -\cos\frac{\pi}{n}\right) \forall 0 \leq \theta < \frac{\pi}{2n}$.
Hence,

$$\begin{aligned}
&p\,(n \equiv 1\,(mod\,4)) \\
=&1 - \int_0^{\frac{2\pi}{n}} \int_{-\min_i V_{r_i x}}^{\max\left(-\min_i V_{r_i x}, w - \max_i V_{r_i x}\right)} \int_{-\min_i V_{r_i y}}^{\max\left(-\min_i V_{r_i y}, h - \max_i V_{r_i y}\right)} \frac{n}{2\pi wh}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\theta \\
=&1 - \int_0^{\frac{\pi}{2n}} \int_{-\cos\left(\theta + \frac{2\pi(n-1)}{2n}\right)}^{\max\left(-\cos\left(\theta+\frac{2\pi(n-1)}{2n}\right), w - \cos\theta\right)} \cdot \\
&\qquad \cdot \int_{-\sin\left(\theta+\frac{2\pi(3n+1)}{4n}\right)}^{\max\left(-\sin\left(\theta+\frac{2\pi(3n+1)}{4n}\right), h - \sin\left(\theta + \frac{2\pi(3n+1)}{4n}\right)\right)} \frac{2n}{\pi wh}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\theta
\end{aligned}$$

(By Lemma 2.6 and 2.9)
When $w, h > 2$,

$$\begin{aligned}
&p\,(n \equiv 1\,(mod\,4)) \\
=&1-\frac{2n}{\pi wh}\int_0^{\frac{\pi}{2n}}\left(w-\cos\theta+\cos\left(\theta+\frac{2\pi(n-1)}{2n}\right)\right) \\
&\qquad\left(h-\sin\left(\theta+\frac{2\pi(n-1}{4n}\right)+\sin\left(\theta+\frac{2\pi(3n+1)}{4n}\right)\right)\mathrm{d}\theta \\
=&\frac{\sin\frac{\pi}{2n}\left(4n(h-w)\cos\frac{\pi}{2n}+4nw-\pi\sin\frac{\pi}{n}\right)}{\pi hw}.
\end{aligned}$$

Case 3: $n \equiv 2\,(mod\,4)$

Note that in this case $\max_i V_{r_i y}$ and $\min_i V_{r_i y}$ both have two possible candidates initially. Like Case 2, we can take ones with smaller indices, meaning $\max_i V_{r_i x}$, $\max_i V_{r_i y}$, $\min_i V_{r_i x}$, $\min_i V_{r_i y} = V_{r_0 x}, V_{r_{\frac{n-2}{4}} y}, V_{r_{\frac{n}{2}} x}, V_{r_{\frac{3n-2}{4}}} = \cos\theta$, $\sin\left(\frac{2\pi(n-2)}{4n}+\theta\right), \cos\left(\frac{2\pi n}{2n}+\theta\right), \sin\left(\frac{2\pi(3n-2)}{4n}+\theta\right)$ respectively. Like Case 1, the probability is

$$\begin{aligned}
&p\,(n \equiv 2\,(mod\,4)) \\
=&1-\int_0^{\frac{\pi}{n}}\int_{-\cos\left(\frac{2\pi n}{2n}+\theta\right)}^{\max\left(-\cos\left(\frac{2\pi n}{2n}+\theta\right),w-\cos\theta\right)}\cdot \\
&\qquad\cdot\int_{-\sin\left(\frac{2\pi(3n-2)}{4n}+\theta\right)}^{\max\left(-\sin\left(\frac{2\pi(3n-2)}{4n}+\theta\right),h-\sin\left(\frac{2\pi(3n-2)}{4n}+\theta\right)\right)}\frac{n}{\pi wh}\,\mathrm{d}y\,\mathrm{d}x\,\mathrm{d}\theta
\end{aligned}$$

Example 2.12 (The Buffon-Laplace Needle Problem). *Given $n = 2$, it falls into case 3 as $n \equiv 2 \pmod 4$.*
Since the l in the original problem is 2 times the radius of the 2 sided polygon,

$$h := \frac{2h}{l}, w := \frac{2w}{l}$$

$l < a, b$ in the original problem, so $h, w > 2$.
Using the formula,

$$p = \frac{2(2)l(2h+2w-l)\sin\frac{\pi}{2} - l^2 2\pi\cos\frac{\pi}{2}}{4\pi wh}$$

$$p = \frac{2l}{\pi w} + \frac{2l}{\pi h} - \frac{l^2}{\pi wh}$$

.

This also agrees with the formula given by Laplace.

Case 4: $n \equiv 3 \pmod 4$

Like previous cases, $\max_i V_{r_{i_x}}, \max_i V_{r_{i_y}}, \min_i V_{r_{i_x}}, \min_i V_{r_{i_y}} = V_{r_{0_x}}, V_{r_{\frac{n+1}{4}_y}}, V_{r_{\frac{n-1}{2}_x}}, V_{r_{\frac{3n-1}{4}_y}} = \cos\theta, \sin\left(\frac{\pi(n+1)}{2n}+\theta\right), \cos\left(\frac{\pi(n-1)}{n}+\theta\right), \sin\left(\frac{\pi(3n-1)}{2n}+\theta\right)$.

Hence,

$$\begin{aligned} &p\left(n \equiv 3 \pmod 4\right) \\ &= 1 - \int_0^{\frac{\pi}{2n}} \int_{-\cos\left(\frac{\pi(n-1)}{n}+\theta\right)}^{\max\left(-\cos\left(\frac{\pi(n-1)}{n}+\theta\right), w-\cos\theta\right)} \cdot \\ &\qquad \cdot \int_{-\sin\left(\frac{\pi(3n-1)}{2n}+\theta\right)}^{\max\left(-\sin\left(\frac{\pi(3n-1)}{2n}+\theta\right)\cdot h-\sin\left(\frac{\pi(n+1)}{2n}+\theta\right)\right)} \frac{2n}{\pi wh} dydxd\theta \end{aligned}$$

When $w, h \geq 2$,

$$\begin{aligned} p &= 1 - \int_0^{\frac{\pi}{2n}} \int_{-\cos\left(\frac{\pi(n+1)}{n}+\theta\right)}^{w-\cos\theta} \int_{-\sin\left(\frac{\pi(3n-1)}{2n}+\theta\right)}^{h-\sin\left(\frac{\pi(n-1)}{2n}+\theta\right)} \frac{2n}{\pi wh} dydxd\theta \\ &= -\frac{2\left(hn\sin\left(\frac{\pi}{n}\right)+\cos\left(\frac{\pi}{2n}\right)\left((1-2h)n\sin\left(\frac{\pi}{n}\right)+\pi\cos\left(\frac{\pi}{n}\right)\right)-2nw\sin\left(\frac{\pi}{2n}\right)\right)}{\pi hw} \end{aligned}$$

3. An Algorithmic Solution to the Arbitrary Polygon Problem

Suppose on a 2D plane, there is a rectangular grid with each rectangle of width w and height, now given a polygon with n sides with each vertex located 1 unit away from the centre of the polygon, if we drop the polygon randomly on the plane, what is the probability p that the polygon intersects with one or more grid lines?

We will be using integration on joint probability density functions[PD] as our main method of finding the probability p.

Let's first consider the case $n = 4$, as a demonstration of the method.

Given an arbitrary polygon Q with n vertices, if it is not a convex polygon, we can convert it into a convex polygon, by changing Q into its convex hull. A convex hull of a shape is the smallest convex shape that contains it. The convex hull may be visualized as the shape enclosed by a rubber band stretched around the shape.

Consider Q rotating clockwise about its geometric centre. It can be seen as the rectangular grid lines being rotated anticlockwise about the centre of Q. It is because rotations about the same centre are relative.

For each rotation of grid lines by θ $(0 \leq \theta < \frac{\pi}{2})$, we need to find the equivalent rectangle of Q. The equivalent rectangle is defined as the smallest rectangle that bounds Q, with its sides parallel to the rotated grid lines. Then, the probability of not intersecting the grid lines for that specific θ is,

$$\frac{A}{wh}$$

where A is the area of the places that can place the equivalent rectangle without intersecting with the grid lines.

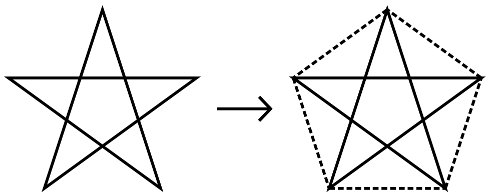


FIGURE 8. Demonstrating a convex hull of a shape.

The range $0 \leq \theta < \frac{\pi}{2}$, is because the equivalent rectangle considers 4 sides of Q at once, so θ is less than $\frac{2\pi}{4} = \frac{\pi}{2}$.

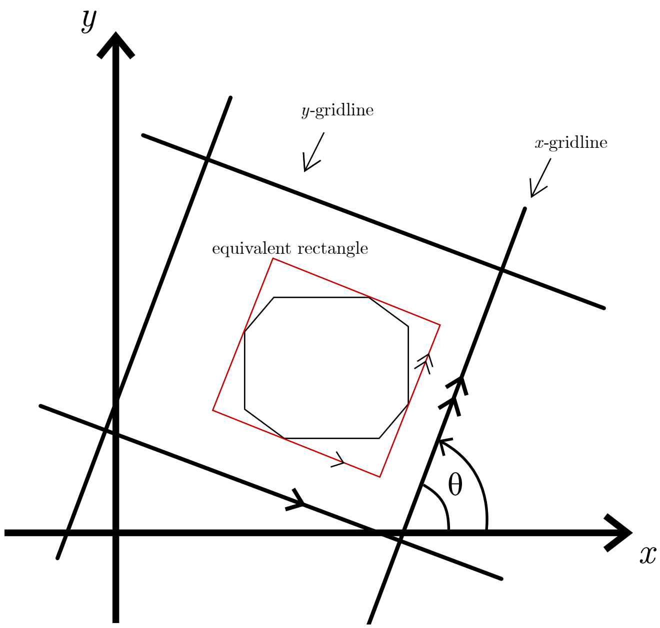


FIGURE 9. A demonstration of rotating the grid lines and equivalent rectangle.

To find the equivalent rectangle of Q for a specific θ, we need to find vertices which are the maximum and minimum values in the rotated x (x') and rotated y (y') directions. We can compare the slopes of the two edges connecting to the same vertex with the gridlines.

Definition 3.1. *Let the first vertex of Q be an arbitrary point on Q. Let the $(i+1)$-th vertex be the next vertex of the i-th vertex in the anticlockwise direction, $\forall\ i \in$*

$\{1, 2, 3, \ldots, n\}$. *Since the vertices in Q is cyclic, the 0-th and $(n+1)$-th vertex/edge are the n-th and 1-st vertex/edge respectively. Let x_i, y_i be the x-coordinate and y-coordinate of i-th vertex in Q. Let i-th edge in Q be the edge connecting the i-th and $(i+1)$-th vertices. Let m_i be the slope of the i-th edge,*

$$m_i = \frac{y_{i+1} - y_i}{x_{i+1} - x_i} \forall i \in \{1, 2, 3, \ldots, n\}$$

For each edge in Q, we can calculate their angle from the x-axis θ_i, by

$$\theta_i = \tan^{-1}(m_i), -\frac{\pi}{2} < \theta_i \leq \frac{\pi}{2}$$

If m_i is undefined, or equivalently when the i-th edge is a vertical line, then $\theta_i = \frac{\pi}{2}$.

Definition 3.2. *$x'_{\max}, x'_{\min}$ are the vertices that touches the x' grid line, with the maximum and minimum x-coordinate respectively. Similarly, $y'_{\max}$, $y'_{\min}$ are the vertices that touches the y' grid line, with the maximum and minimum y-coordinate respectively.*

Lemma 3.3. *If the i-th vertex is $x'_{\max}$ or $x'_{\min}$, then $\theta_{i-1} \leq \theta < \theta_i$. Similarly, if the i-th vertex is $y'_{\max}$ or $y'_{\min}$, then $\theta_{i-1} + \frac{\pi}{2} \leq \theta < \theta_i + \frac{\pi}{2}$.*

Proof. We will demonstrate the proof with the use of diagrams.

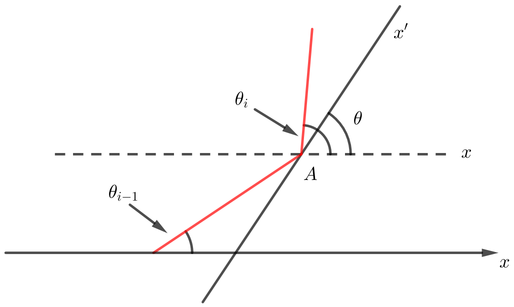


FIGURE 10.

In Figure 10, A is the i-th vertex of Q, and the red lines are the edges of Q. A is $x'_{\max}$ or $x'_{\min}$. It can be seen that $\theta_{i-1} \leq \theta < \theta_i$. The reason why θ is strictly less than θ_i is that if $\theta = \theta_i$, the $(A+1)$-th vertex is considered as $x'_{\max}$ or $x'_{\min}$ instead of A.

Similarly, in Figure 11, A is the i-th vertex of Q, and the red lines are the edges of Q. A is $y'_{\max}$ or $y'_{\min}$.

It can be seen that $-\theta_{i-1} \geq \frac{\pi}{2} - \theta > -\theta_i \Leftrightarrow \theta_{i-1} + \frac{\pi}{2} \leq \theta < \theta_i + \frac{\pi}{2}$.

The reason why θ is strictly less than $\theta_i + \frac{\pi}{2}$ is that if $\theta = \theta_i$, the $(A+1)$-th vertex is considered as $y'_{\max}$ or $y'_{\min}$ instead of A.

$\square$

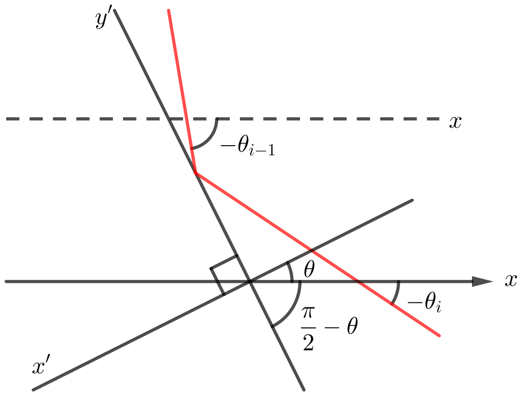


FIGURE 11.

We need to keep track of the angle θ when either one of $y'_{\max}, y'_{\min}, x'_{\max}, x'_{\min}$ changes. Therefore, we will introduce a new term, critical angle.

Definition 3.4. *The critical angle (x) for the i-th vertex in Q, which is denoted by C_{x_i}, is the angle θ when either one of $x'_{\max}$ and $x'_{\min}$ changes from the i-th vertex to the $(i+1)$-th vertex. Similarly, the critical angle (y) for the i-th vertex in Q, which is denoted by C_{y_i}, is the angle θ when either one of $y'_{\max}$ and $y'_{\min}$ changes from the i-th vertex to the $(i+1)$-th vertex.*

Using Lemma 3.3, $x'_{\max}$ or $x'_{\min}$ changes from the i-th vertex to the $(i+1)$-th vertex when $\theta = \theta_i$, and $y_{\max}$ or $y_{\min}$ changes from the i-th vertex to the $(i+1)$-th vertex when $\theta = \theta_i + \frac{\pi}{2}$. Therefore,

$$C_{x_i} = \theta_i$$

$$C_{y_i} = \theta_i + \frac{\pi}{2}$$

.

Formula 3.5. *Suppose that from $\theta = \theta_{\text{now}}$ to $\theta = \theta_{\text{next}}$, any of $y'_{\max}, y'_{\min}, x'_{\max}, x'_{\min}$ will not change. Suppose $d_x \leq w, d_y \leq h \ \forall \theta \in [\theta_{now}, \theta_{next})$. Let the coordinates of $y'_{\max}, y'_{\min}, x'_{\max}, x'_{\min}$ be $(p_1, q_1), (p_2, q_2), (p_3, q_3), (p_4, q_4)$ respectively, then the continuous sum of probabilities of not intersecting the grid lines when $\theta \in [\theta_{now}, \theta_{next})$ is:*

$$(\theta_{next} - \theta_{now}) - \frac{d_{a,b}}{w}\left(\int_{\theta_{now}}^{\theta_{next}} |\sin(\alpha+\theta)|d\theta\right) - \frac{d_{c,d}}{h}\left(\int_{\theta_{now}}^{\theta_{next}} |\sin(\beta-\theta)|d\theta\right)$$
$$+\frac{d_{a,b}d_{c,d}}{2wh}\int_{\theta_{now}}^{\theta_{next}} |\cos(\alpha-\beta+2\theta)-\cos(\alpha+\beta)|d\theta$$

The variables used in this formula will be defined below.

Definition 3.6. *For a specific θ, define A as the area of space that can place the equivalent rectangle without intersecting the grid lines. Let d_x, d_y be the distance between $x'_{\max}$, $x'_{\min}$ which is parallel to x', and the distance of $y'_{\max}$, $y'_{\min}$ which is parallel to y' respectively.*

$$A = wh - d_x h - d_y w + d_x d_y$$
$$d_x = \sqrt{a^2+b^2}|\sin(\alpha+\theta)|$$
$$d_y = \sqrt{c^2+d^2}|\sin(\beta-\theta)|$$
$$a = p_1 - p_2$$
$$b = q_1 - q_2$$
$$c = p_3 - p_4$$
$$d = q_3 - q_4$$
$$\tan\alpha = \frac{a}{b}$$
$$\tan\beta = \frac{d}{c}$$

Proof of formula 3.5. We can observe that the dimensions of the equivalent rectangle are $d_x \times d_y$. We will once again demonstrate the proof with the use of diagrams. From Figure 12 and 13,

$$|\sin(\alpha+\theta)| + \frac{d_X}{d_{a,b}}$$

where $d_{a,b} = \sqrt{a^2+b^2}$,

$$d_x = d_{a,b}|\sin(\alpha+\theta)|$$

$$dx = \begin{cases} d_{a,b}\sin(\alpha+\theta), & \text{if}\, \theta > -\alpha \\ -d_{a,b}\sin(\alpha+\theta), & \text{otherwise} \end{cases}$$

Similarly

$$|\sin(\beta-\theta)| = \frac{d_y}{d_{c,d}}$$

where $d_{c,d} = \sqrt{c^2+d^2}$,

$$d_y = d_{c,d}|\sin(\beta-\theta)|$$

$$d_y = \begin{cases} d_{c,d}\sin(\beta-\theta), & \text{if}\, \theta < \beta \\ -d_{a,b}\sin(\beta-\theta), & \text{otherwise} \end{cases}$$

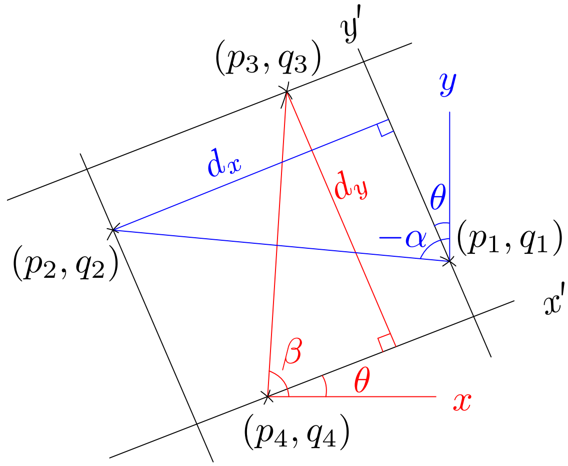


FIGURE 12.

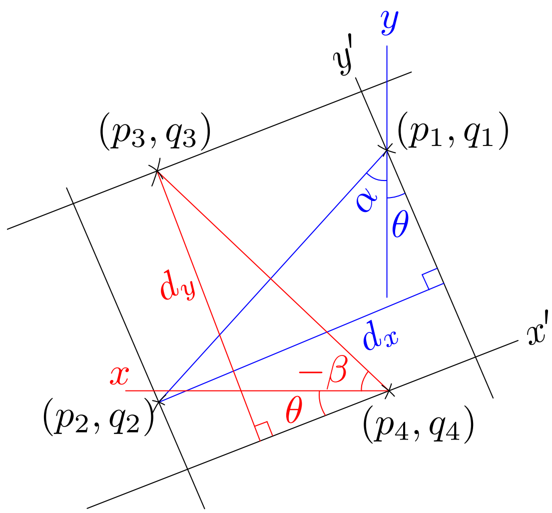


FIGURE 13.

Thus,

$$d_x d_y = d_{a,b} d_{c,d} |\sin(\alpha+\theta)\sin(\beta-\theta)|$$
$$d_x d_y = \frac{d_{a,b} d_{c,d}}{2} |\cos(\alpha-\beta+2\theta) - \cos(\alpha+\beta)|$$

Therefore

$$A = \max((w-d_x),0)\cdot\max((h-d_y),0)$$

As $d_x \leq w$, $d_y \leq h$, $\forall\theta \in [\theta_{\text{now}}, \theta_{\text{next}})$,

$$A = wh - d_x h - d_y w + d_x d_y$$

Therefore, the continuous sum of probability of not intersecting the grid lines when $\theta \in [\theta_{\text{now}}, \theta_{\text{next}})$ is

$$\begin{aligned}
\int_{\theta_{\text{now}}}^{\theta_{\text{next}}} \frac{A}{wh} d\theta &= \frac{1}{wh} \int_{\theta_{\text{now}}}^{\theta_{\text{next}}} (wh - d_x h - d_y w + d_x d_y) d\theta \\
&= (\theta_{\text{next}} - \theta_{\text{now}}) - \frac{d_{a,b}}{w} \left(\int_{\theta_{\text{now}}}^{\theta_{\text{next}}} |\sin(\alpha + \theta)| d\theta \right) \\
&\quad - \frac{d_{c,d}}{h} \left(\int_{\theta_{\text{now}}}^{\theta_{\text{next}}} |\sin(\beta - \theta) d\theta| \right) \\
&\quad + \frac{d_{a,b} d_{c,d}}{2wh} \int_{\theta_{\text{now}}}^{\theta_{\text{next}}} |\cos(\alpha - \beta + 2\theta) - \cos(\alpha + \beta)| d\theta.
\end{aligned}$$

□

To find the integral of $|\cos(\alpha - \beta + 2\theta) - \cos(\alpha + \beta)|$, let $\cos(\alpha - \beta + 2\theta) = \cos(\alpha + \beta)$, then

$$\theta = \beta + k\pi \text{ or } -\alpha + k\pi, \quad \forall k \in \mathbb{Z}$$

Then find all $\theta \in [\theta_{\text{now}}, \theta_{\text{next}})$. For each θ, we can split the integral, then find the sign of $\cos(\alpha - \eta + 2\theta) - \cos(\alpha + \beta)$ in each part.

Example 3.7. *There is a solution θ_{sol}. When $\theta \in [\theta_{now}, \theta_{next})$, $\cos(\alpha - \eta + 2\theta) - \cos(\alpha + \beta) > 0$. When $\theta \in [\theta_{sol}, \theta_{next})$, $\cos(\alpha - \eta + 2\theta) - \cos(\alpha + \beta) < 0$. Then,*

$$\begin{aligned}
&\int_{\theta_{now}}^{\theta_{next}} |\cos(\alpha - \eta + 2\theta) - \cos(\alpha + \beta)| d\theta \\
&= \frac{1}{2}(\sin(\alpha - \beta + 2t) - \sin(\alpha - \beta + 2s)) - (t - s)\cos(\alpha + \beta)
\end{aligned}$$

Similarly, we can use this method to find the integral of $|\sin(\alpha + \theta)|$ and $|\sin(\beta - \theta)|$.

With all these preparations, we can start writing the algorithm.

3.1. **Algorithm.**

$Q :=$ convex hull of Q
Let $d_{\max} = 0$
for each pair of vertices (i, j) do

$$d_{\max} := \max\left(d_{\max}, \sqrt{(i_x - j_x)^2 + (i_y - j_y)^2}\right)$$

if $d_{\max} > w$ or $d_{\max} > h$ then exit.
Let $\theta := 0$, sum:= 0
while $\theta < \frac{\pi}{2}$ do
    find (v_1, v_2) such that $\theta_{v_{i-1}} + \frac{\pi}{2} \leq \theta \leq \theta_{v_i} + \frac{\pi}{2}$, $\forall i = 1, 2$
    find (v_3, v_4) such that $\theta_{v_{i-1}} \leq \theta \leq \theta_{v_i}$, $\forall i = 3, 4$
    Let $\theta_{critical} = \min\left(C_{y_{v_1}}, C_{y_{v_2}}, C_{x_{v_3}}, C_{x_{v_4}}, \frac{\pi}{2}\right)$

Let A_{current} = result of Formula 3.5 with v_1, v_2, v_3, v_4 as vertices, $\theta_{\text{now}} = \theta, \theta_{\text{next}} = \theta_{\text{critical}}$ as input.

$sum := sum + A_{current}$

$\theta := \theta_{critical}$ end

$p = 1 - \frac{2 \cdot sum}{\pi}$

3.2. **Explanation.** $d_{\max}$ is the maximum of all d_x and d_y for all vertices and $\theta \in [0, \frac{\pi}{2})$.

If $d_{\max} > w, h$ then it may not meet the condition for Formula 3.5, which is $d_x \leq w$, $d_y \leq h$ for some vertices and θ. Therefore, the algorithm cannot compute the correct probability p. The algorithm is then exited.

At the start of each loop,

We find the vertices of $y'_{\max}, y'_{\min}$ as v_1, v_2 respectively, vertices of $x'_{\max}, x'_{\min}$ as v_3, v_4 respectively, by Lemma 3.3.

Then we find the angle which one of $y'_{\max}, y'_{\min}, x'_{\max}, x'_{\min}$ change, $\theta_{critical}$, which is the minimum of the critical angles (x) of v_1, v_2, and the critical angles (y) of v_3, v_4. Moreover, it cannot be greater than $\frac{\pi}{2}$ since $0 \leq \theta < \frac{\pi}{2}$. Therefore, we also add $\frac{\pi}{2}$ as a parameter in the min function.

After that, we calculate the continuous sum of the probability by Formula 3.5 with v_1, v_2, v_3, v_4 being vertices, $\theta_{now} = \theta$, $\theta_{next} = \theta_{critical}$, then add it to our sum variable.

We then set θ to be $\theta_{critical}$, and continue executing the loop while $\theta < \frac{\pi}{2}$.

The probability that Q does not intersect with the grid lines is the sum of the continuous sum of probabilities multiplied by the probability density function of θ, which is $\frac{2}{\pi}$.

$$P(\text{no intersection}) = \frac{2 \cdot sum}{\pi}$$

The final probability p that Q intersects with the grid lines is

$$p = 1 - P(\text{no intersection})$$

$$p = 1 - \frac{2 \cdot sum}{\pi}$$

3.3. **Time complexity.** At the start, we will find the convex hull of Q, it requires $O(n \log n)$ time for using Graham scan algorithm[GS].

Then, we will find that maximum distance between every pair of vertices. This can be done using a two-pointer algorithm [2P] in $O(n')$, where n' is the number of vertices of the convex hull of Q.

$y'_{\max}, y'_{\min}, x'_{\max}, x'_{\min}$ will change a total of $4n'$ times at maximum. When any of $y'_{\max}, y'_{\min}, x'_{\max}, x'_{\min}$ changes, it will change to the next vertex in the anti-clockwise direction. Therefore, each of them will change its vertex n' times, and the loop will be executed $O(n')$ times.

In each iteration of the loop, it will find new vertices for $y'_{\text{max}}, y'_{\text{min}}, x'_{\text{max}}, x'_{\text{min}}$. We only need to check the next vertex in the anticlockwise direction. This require $O(n')$ for the first iteration, and constant time in the following iterations. The rest in the loop can be computed in $O(1)$.

The time complexity for the loop is $O(n') \cdot O(1) = O(n')$. Therefore, the final time complexity is $O(n \log n)$.

Example 3.8. *Let Q be a regular polygon with 8 sides, with the distance between each vertex and the geometric centre of the polygon being ℓ units. What is the probability p that when Q is dropped randomly to a rectangular grid with width w and height h, intersects with one or more grid line?*

We can define the i-th vertex $\left(\ell\cos(\frac{\pi i}{4}), \ell\sin(\frac{\pi i}{4})\right)$.
We can observe that all of $y'_{\text{max}}, y'_{\text{min}}, x'_{\text{max}}, x'_{\text{min}}$ *change their vertices when* $\theta = \frac{\pi}{8}, \frac{3\pi}{8}, \frac{\pi}{2}$.

Therefore, the loop will be executed 3 *times.*

- *First iteration* ($\theta_{now} = 0, \theta_{next} = \frac{\pi}{8}$)
  Find $v_1, v_2, v_3, v_4 = 0, 4, 2, 6$
  Using Formula 3.5 $A_{current} = \frac{\pi}{8} - 2(\frac{\ell}{w} + \frac{\ell}{h})\sin(\frac{\pi}{8}) + \frac{\ell^2}{wh}(\frac{\sqrt{2}}{2} + \frac{\pi}{4})$
- *Second iteration* ($\theta_{now} = \frac{\pi}{8}, \theta_{next} = \frac{3\pi}{8}$)
  Find $v_1, v_2, v_3, v_4 = 1, 5, 3, 7$
  Using Formula 3.5 $A_{current} = \frac{\pi}{4} - 4(\frac{\ell}{w} + \frac{\ell}{h})\sin(\frac{\pi}{8}) + \frac{\ell^2}{wh}(\sqrt{2} + \frac{\pi}{2})$
- *Third iteration* ($\theta_{now} = \frac{3\pi}{8}, \theta_{next} = \frac{\pi}{2}$)
  Find $v_1, v_2, v_3, v_4 = 2, 6, 4, 0$
  Using Formula 3.5 $A_{current} = \frac{\pi}{8} - 2(\frac{\ell}{w} + \frac{\ell}{h})\sin(\frac{\pi}{8}) + \frac{\ell^2}{wh}(\dfrac{\sqrt{2}}{2} + \frac{\pi}{4})$

After all iterations,

$$sum = \frac{\pi}{2} - 8\left(\frac{\ell}{w} + \frac{\ell}{h}\right)\sin\left(\frac{\pi}{8}\right) + \frac{\ell^2}{wh}\left(2\sqrt{2} + \pi\right)$$

Therefore

$$p = \frac{16}{\pi}\left(\frac{1}{w} + \frac{1}{h}\right)\sin\left(\frac{\pi}{8}\right) - \frac{2\ell^2}{wh} - \frac{4\sqrt{2}\ell^2}{\pi wh}$$

This result also agrees with Example 2.11.

4. Dropping Right Regular Prisms in Cuboidal Grid

Suppose in a 3*D space, there is a cuboidal grid with each cuboid of length ℓ, width w and height h. Each face of the cuboid forms its corresponding grid plane. Given a right regular prism of height k with* 3*n edges with each vertex located 1 unit away from the nearest centre of the n-sided regular polygon base, if we drop the prism* *randomly in the space, what is the probability p that the prism intersects with one or more grid plane(s) ?*

Definition 4.1. *Given a $n-$sided polygon placed in $3D$ space. The size of the polygon is said to be default if each vertex of the polygon is located 1 unit away from the geometric centre of the polygon. The position of the polygon is said to be* ***default*** *when its geometric centre is located at the origin. The orientation of the polygon is said to be* ***default*** *when 1 of its vertices is located at $(0,0,1)$ and the polygon lies on the xz-plane. If the size, position and orientation of the polygon are default, let the vertex located at $(0,1)$ be the zeroth vertex V_0 and the $i-$th vertex V_i be the next vertex V_{i-1} in the anticlockwise direction, $\forall i \in \{1,2,3,\ldots,n-1\}$.Otherwise, let the zeroth vertex V_0 of the polygon be an arbitrary vertex of the polygon and the $i-$th vertex V_i be the next vertex of V_{i-1} in the anticlockwise direction, $\forall i \in \{\ 1,2,3,\ldots,n-1\}$. Since the vertices in the polygon are cyclic, the (n+1)-th vertex is the 1-st vertex. Let $x_{\max}, x_{\min}, y_{\max}, y_{\min}, z_{\min}, z_{\max}$ be the largest x-coordinate, the smallest x-coordinate, the largest y-coordinate, the smallest y-coordinate, the largest z-coordinate and the smallest z-coordinate of the vertices of the polygon among the n vertices respectively.*

In a n-sided regular polygon with default size, position and orientation, the coordinates of V_i are given by $\left(-\sin\frac{2\pi}{n}, 0, \cos\frac{2\pi i}{n}\right)$

Lemma 4.2. *Given a n-sided polygon placed in 3D space with default position and all vertices being equidistant from the geometric centre of the polygon. Among the n vertices of the polygon, the vertices whose connections with the origin make the smallest angles with the positive x-axis, the negative x-axis, the positive y-axis, the negative y-axis, the positive z-axis and the negative z-axis have the largest x-coordinate, the smallest x-coordinate, the largest y-coordinate, the smallest y-coordinate, the largest z-coordinate and the smallest z-coordinate respectively.*

Proof of Lemma 4.2. Let $\theta_{n,V_i,\text{positive } x\text{-axis}}$, $\theta_{n,V_i,\text{negative } x\text{-axis}}$, $\theta_{n,V_i,\text{positive } y\text{-axis}}$, $\theta_{n,V_i,\text{negative } y\text{-axis}}$, $\theta_{n,V_i,\text{positive } z\text{-axis}}$, $\theta_{n,V_i,\text{negative } z\text{-axis}}$ be the angle between V_iO and the positive x-axis, between V_iO and the negative x-axis,between V_iO and the positive y-axis, between V_iO and the negative y-axis, between V_iO and the positive z-axis, between V_iO and the negative z-axis respectively for all $i \in \{0,1,\cdots,n-1\}$. Let $V_{i_{x_{\max}}}$, $V_{i_{x_{\min}}}$, $V_{i_{y_{\max}}}$, $V_{i_{y_{\min}}}$, $V_{i_{z_{\max}}}$, $V_{i_{z_{\min}}}$ be the vertices of the polygon whose connections with the origin make the smallest angle with the positive x-axis, the negative x-axis, the positive y-axis, the negative y-axis, the positive z-axis and the negative z-axis respectively. Let s be the length of V_iO.

The x-coordinate of $V_i = s\cos\pm\theta_{n,V_i,\text{positive } x\text{-axis}}$ which is strictly decreasing for $\theta_{n,V_i,\text{positive } x\text{-axis}} \in (0,\pi)$.

Since $\theta_{n,V_i,\text{positive } x\text{-axis}} \geq \theta_{n,V_{i_{x_{\max}}},\text{positive } x\text{-axis}}$, the x-coordinate of $V_i \leq$ the x-coordinate of $V_{i_{x_{\max}}}$.

The x-coordinate of $V_i = s\cos(\pi \pm \theta_{n,V_i,\text{negative } x\text{-axis}}) = -s\cos\theta_{n,V_i,\text{negative } x\text{-axis}}$ which is strictly increasing for $\theta_{n,V_i,\text{negative } x\text{-axis}} \in (0,\pi)$.

Since $\theta_{n,V_i,\text{negative } x\text{-axis}} \geq \theta_{n,V_{i_{x_{\min}}},\text{positive } x\text{-axis}}$, the x-coordinate of $V_i \geq$ the x-coordinate of $V_{i_{x_{\min}}}$.

The y-coordinate of $V_i = s\cos \pm\theta_{n,V_i,\text{positive } y\text{-axis}}$ which is strictly decreasing for $\theta_{n,V_i,\text{positive } y\text{-axis}} \in (0,\pi)$.

Since $\theta_{n,V_i,\text{positive } y\text{-axis}} \geq \theta_{n,V_{i_{y_{\max}}},\text{positive } y\text{-axis}}$, the y-coordinate of $V_i \leq$ the y-coordinate of $V_{i_{y_{\max}}}$.

The y-coordinate of $V_i = s\cos(\pi \pm \theta_{n,V_i,\text{negative } y\text{-axis}}) = -s\cos\theta_{n,V_i,\text{negative } y\text{-axis}}$ which is strictly increasing for $\theta_{n,V_i,\text{negative } y\text{-axis}} \in (0,\pi)$.

Since $\theta_{n,V_i,\text{negative } y\text{-axis}} \geq \theta_{n,V_{i_{y_{\min}}},\text{positive } y\text{-axis}}$, the y-coordinate of $V_i \geq$ the y-coordinate of $V_{i_{x_{\min}}}$.

The z-coordinate of $V_i = s\cos \pm\theta_{n,V_i,\text{positive } z\text{-axis}}$ which is strictly decreasing for $\theta_{n,V_i,\text{positive } z\text{-axis}} \in (0,\pi)$.

Since $\theta_{n,V_i,\text{positive } z\text{-axis}} \geq \theta_{n,V_{i_{z_{\max}}},\text{positive } z\text{-axis}}$, the z-coordinate of $V_i \leq$ the z-coordinate of $V_{i_{z_{\max}}}$.

The z-coordinate of $V_i = s\cos(\pi \pm \theta_{n,V_i,\text{negative } z\text{-axis}}) = -s\cos\theta_{n,V_i,\text{negative } z\text{-axis}}$ which is strictly increasing for $\theta_{n,V_i,\text{negative } z\text{-axis}} \in (0,\pi)$.

Since $\theta_{n,V_i,\text{negative } z\text{-axis}} \geq \theta_{n,V_{i_{z_{\min}}},\text{positive } z\text{-axis}}$, the z-coordinate of $V_i \geq$ the z-coordinate of $V_{i_{z_{\min}}}$.

Therefore, the vertices whose connections with the origin make the smallest angles with the positive x-axis, the negative x-axis, the positive y-axis, the negative y-axis, the positive z-axis and the negative z-axis have the largest x-coordinate, the smallest x-coordinate, the largest y-coordinate, the smallest y-coordinate, the largest z-coordinate and the smallest z-coordinate respectively. $\square$

Given a n-sided polygon which lies on the xy-plane, xz-plane or yz-plane in 3D space with default position and all vertices being equidistant from the geometric centre of the polygon. Using Lemma 4.2, we can now deduce that the vertices of the polygon carrying the values of $x_{\max}$, $x_{\min}$,$y_{\max}$, $y_{\min}$,$z_{\max}$, $z_{\min}$ do not change if the respective angles between the line segments joining these vertices and the origin, and the positive x-axis, the negative x-axis, the positive y-axis, the negative y-axis, the positive z-axis and the negative z-axis are the smallest among angles between V_iO and the corresponding axes, $\forall i \in \{0, 1, \ldots, n-1\}$.

Therefore, we can deduce that the rotation of a rectangle of width w, height h with $V_0 = \left(\frac{w}{2}, 0, \frac{h}{2}\right)$, $V_1 = \left(-\frac{w}{2}, 0, \frac{h}{2}\right)$, $V_2 = \left(-\frac{w}{2}, 0, -\frac{h}{2}\right)$ and $V_3 = \left(\frac{w}{2}, 0, -\frac{h}{2}\right)$ respectively rotating about the x-axis, y-axis and z-axis by less than $\frac{\pi}{2}, \frac{\pi}{2} and \frac{\pi}{2}$ anti-clockwise or clockwise respectively does not change the vertices carrying the values of $x_{max}, x_{min}, y_{max}, y_{min}, z_{max}, z_{min}$.

Given a n-sided regular polygon with default position and orientation. Using Lemma 4.2 and observing that $V_{\lfloor \frac{n+1}{4} \rfloor}O$, $V_{\lceil \frac{n-1}{2} \rceil}O$, $V_{n-\lfloor \frac{n+2}{4} \rfloor}O$ make the smallest angle with the negative x-axis, the negative y-axis and the positive x-axis respectively among all line segments joining V_i, $\forall i \in \{0, 1, \ldots, n\}$ with the origin, we can deduce that the coordinates of the vertices of the n-sided regular polygon with default position and orientation carrying the respective values of $x_{min}, y_{min}, x_{max}$ are given by,

$$\left(-\sin\frac{2\pi i_{x_{\min}}(n)}{n}, \cos\frac{2\pi i_{x_{\min}}(n)}{n}\right),$$

$$\left(-\sin\frac{2\pi i_{y_{\min}}(n)}{n}, \cos\frac{2\pi i_{y_{\min}}(n)}{n}\right),$$

$$\left(-\sin\frac{2\pi i_{x_{\max}}(n)}{n}, \cos\frac{2\pi i_{x_{\max}}(n)}{n}\right),$$

where $i_{x_{\min}}(n) = \lfloor \frac{n+1}{4} \rfloor$, $i_{y_{\min}}(n) = \lceil \frac{n-1}{2} \rceil$,$i_{x_{\max}}(n) = n - \lfloor \frac{n+1}{4} \rfloor$

Definition 4.3. *Given a right regular prism with a height of k with each vertex located 1 unit away from the nearest centre of the n-sided regular polygon base placed in $3D$ space. The position of the prism is said to be **default** when its geometric centre is located at the origin. The orientation of the prism is said to be **default** when the 2 bases of the prism are parallel to the yz-plane and 2 of its vertices are located at $\left(\frac{k}{2}, 0, 1\right)$ and $\left(-\frac{k}{2}, 0, 1\right)$ respectively. Let $V_{x_{max}}, V_{x_{max}}, V_{y_{max}}, V_{y_{min}}, V_{z_{max}}, V_{z_{min}}$ be the vertices carrying the values of $x_{max}, x_{min}, y_{max}, y_{min}, z_{max}, z_{min}$ respectively.*

Consider a right regular prism with default position and orientation with its 2 bases being n-sided regular polygon. Within the range of $-\frac{\pi}{n} \le \alpha \le \frac{\pi}{n}$, $0 \le \beta, \gamma \le \frac{\pi}{2}$, the vertices carrying the respective values of $x_{max}, x_{min}, y_{max}, y_{min}, z_{max}, z_{min}$ do not change.

By observation,

$$V_{x_{max},x} = \frac{k}{2},\ V_{x_{max},y} = -\frac{1}{2}\sin\frac{2\pi i_{y_{min}}(n)}{n}\sec\frac{\pi(n-2)}{2n},$$
$$V_{x_{max},z} = \frac{1}{2}\cos\frac{2\pi i_{y_{min}}(n)}{n}\sec\frac{\pi(n-2)}{2n},\ V_{x_{min},x} = -\frac{k}{2},\ V_{x_{min},y} = 0,$$
$$V_{x_{min},x} = \frac{1}{2}\sec\frac{\pi(n-2)}{2n},\ V_{y_{max},x} = \frac{k}{2},\ V_{y_{max},y} = -\frac{1}{2}\sin\frac{2\pi i_{x_{max}}(n)}{n}\sec\frac{\pi(n-2)}{2n}$$
$$V_{y_{max},z} = \frac{1}{2}\cos\frac{2\pi i_{x_{max}}(n)}{n}\sec\frac{\pi(n-2)}{2n},\ V_{y_{min},x} = -\frac{k}{2},$$
$$V_{y_{min},y} = -\frac{1}{2}\sin\frac{2\pi i_{x_{min}}(n)}{n}\sec\frac{\pi(n-2)}{2n}, V_{z_{max},x} = \frac{k}{2}, V_{z_{max},y} = 0$$
$$V_{z_{max},z} = \frac{1}{2}\sec\frac{\pi(n-2)}{2n},\ V_{z_{min},x} = -\frac{k}{2}, V_{z_{min},y} = -\frac{1}{2}\sin\frac{2\pi i_{y_{min}}(n)}{n}\sec\frac{\pi(n-2)}{2n},$$
$$V_{z_{min},z} = \frac{1}{2}\cos\frac{2\pi i_{y_{min}}(n)}{n}\sec\frac{\pi(n-2)}{2n}$$

The rotation of the vertices can be described by multiplying them with the 3D rotation matrix [RM]

$$\begin{aligned} R &= R_z(\gamma)R_y(\beta)R_x(\alpha) \\ &= \begin{bmatrix} \cos\gamma & -\sin\gamma & 0 \\ \sin\gamma & \cos\gamma & 0 \\ 0 & 0 & 1 \end{bmatrix} \begin{bmatrix} \cos\beta & 0 & \sin\beta \\ 0 & 1 & 0 \\ -\sin\beta & 0 & \cos\beta \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 0 & \cos\alpha & -\sin\alpha \\ 0 & \sin\alpha & \cos\alpha \end{bmatrix}, \end{aligned}$$

which rotates the vertices about the z-axis by γ anticlockwise, then about the y-axis by β anticlockwise and finally about the x-axis by α anticlockwise.
Therefore, for a specific rotation (γ, β, α), the dimensions of the equivalent cuboid are

$$(d_x, d_y, d_z) =$$

$$(|(RV_{x_{max}})_x - (RV_{x_{min}})_x|, |(RV_{y_{max}})_y - (RV_{y_{min}})_y|, |(RV_{z_{max}})_z - (RV_{z_{min}})_z|)$$

Then, the volume of space free A for placing the equivalent cuboid is

$$A = \max((l - d_x), 0)\max((w - d_y), 0)\max((h - d_x), 0).$$

The joint probability density function of $x, y, z, \alpha, \beta, \gamma$ is

$$\frac{1}{l} \cdot \frac{1}{w} \cdot \frac{1}{h} \cdot \left(\frac{2}{\pi}\right)^2 \cdot \frac{n}{2\pi} = \frac{2n}{lwh\pi^3}.$$

Thus, the probability p that the prism intersects with one or more grid planes is

$$p = 1 - \frac{2n}{lwh\pi^3}\int_0^{\frac{\pi}{2}}\int_0^{\frac{\pi}{2}}\int_{-\frac{\pi}{n}}^{\frac{\pi}{n}} A\, d\alpha d\beta d\gamma$$

5. Generalisation to Higher Dimensions

In an n-dimensional space, denote its axes by $X_1, X_2, \cdots, X_n$. Let the spacing between the hyperplanes be $d_1, d_2, \cdots, d_n$. A point $P = (p_1, p_2, \cdots, p_n)$ is said to be *on the hyperplanes* if $p_i = kd_i$ if some $1 \leq i \leq n$ and $k \in \mathbb{Z}$.

Randomly dropping a needle of length l into this space is equivalent to randomly choosing two points whose Euclidean distance equals l. Let

$$A = (a_1, a_2, \cdots, a_n),\ 0 \leq a_i \leq \frac{d_i}{2}$$

be one end of the needle and

$$B = (b_1, b_2, \cdots, b_n),\ |A - B| = \sqrt{\sum_i (a_i - b_i)^2} = l$$

be another end of the needle. We divide the current hyper-cuboid (the one that A is in) bounded by hyperplanes into 2^n equal hyper-cuboids and work on the one containing the origin. Dropping needles on the other regions are symmetrical to that one, and hence ignoring them would not change the probability. Let A be randomly placed on the hyperspace, meaning a_i are independent random variables with uniform probability distribution functions. Then let B be a point randomly selected on the $(n-1)$-sphere centered at A with radius l. Denote the hypersphere $T^{n-1} = \{x \in \mathbb{R}^n : |x - A^T| = l\}$. This hypersphere is homeomorphic to the normal unit $(n-1)$-sphere, denoted $S^{n-1} = \{x \in \mathbb{R}^n : |x| = 1\}$. We can then establish a bijective mapping $B : S^{n-1} \to T^{n-1}$ to get B.

Let $x_1, x_2, \cdots, x_n$ be n Gaussian random variables and X be $\begin{bmatrix} x_1 \\ x_2 \\ \vdots \\ x_n \end{bmatrix}$. Then the normalised vector $X_n = \frac{X}{|X|}$ is uniformly distributed on the surface S^{n-1}.
The probability density function of x_i we chose is the normal distribution with $\mu = 0$ and $\sigma^2 = 1$, i.e., $\frac{e^{-\frac{x_i^2}{2}}}{\sqrt{2\pi}}$. Their joint probability density function is then

$$\prod_{i=1}^{n} \frac{e^{-\frac{x_i^2}{2}}}{\sqrt{2\pi}}$$

After getting X, we can set $B = l \cdot \frac{X}{|X|} + A$. Let the needle be AB. AB **does not** intersect the hyperplanes when

$$\begin{aligned} & b_i > 0 \\ \Leftrightarrow\ & l \cdot \frac{x_i}{|X|} + a_i > 0 \\ \Leftrightarrow\ & |X| > \frac{l \cdot x_i}{-a_i} \end{aligned} \tag{3.1}$$

Because this involves the norm of X, instead of calculating the norm from X, we calculate x_n from $x_{1\cdots n-1}$ and $|X|$:

$$|X| = \sqrt{x_1^2 + x_2^2 + \cdots + x_n^2}$$

$$\Leftrightarrow x_n = \pm\sqrt{|X|^2 - x_1^2 - x_2^2 - \cdots - x_{n-1}^2} \tag{3.2}$$

$$\Leftrightarrow |X|^2 \geq x_1^2 + x_2^2 + \cdots + x_{n-1}^2. \tag{3.3}$$

We split the calculation into two parts according to (3.2), namely positive x_n and negative x_n.

Case 1: Positive x_n

In this case, $x_n = \sqrt{|X|^2 - x_1^2 - x_2^2 - \cdots - x_{n-1}^2}$. Formula (3.1) is always true when $i = n$. Therefore, we do not need to consider x_n when determining if there is intersection. Therefore, the probability of AB no intersecting the hyperplane when x_n is positive is

Therefore, the probability of the needle **not** intersecting the hyperplanes is

$$\begin{aligned}
&P(\text{not intersect when } x_n \text{ is positive})\\
&= \int_0^{\frac{d_1}{2}} da_1 \cdots \int_0^{\frac{d_n}{2}} da_n \int_{-\infty}^{\infty} dx_1 \cdots \\
&\qquad \cdots \int_{-\infty}^{\infty} dx_{n-1} \int_{\max\left(x_1^2+x_2^2+\cdots+x_{n-1}^2,\ \max\limits_{1\leq i<n} \frac{l\cdot x_i}{-a_i}\right)}^{\infty} \prod_{i=1}^{n} \frac{e^{-\frac{x_i^2}{2}}}{\sqrt{2\pi}} d|X|\\
&= \int_0^{\frac{d_1}{2}} da_1 \cdots \int_0^{\frac{d_n}{2}} da_n \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{n-1} \int_M^{\infty} \frac{e^{-\frac{|X|^2-S}{2}}}{\sqrt{2\pi}} \prod_{i=1}^{n-1} \frac{e^{-\frac{x_i^2}{2}}}{\sqrt{2\pi}} d|X|
\end{aligned}$$

(Let $S = x_1^2 + x_2^2 + \cdots + x_{n-1}^2$ and $M = \max\left(S, \max\limits_{1\leq i<n} \frac{l\cdot x_i}{-a_i}\right)$)

$$= \int_0^{\frac{d_1}{2}} da_1 \cdots \int_0^{\frac{d_n}{2}} da_n \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} dx_{n-1} \int_{\sqrt{M^2-S}}^{\infty} \frac{x_n}{\sqrt{x_n^2+S}} \cdot \prod_{i=1}^{n} \frac{e^{-\frac{x_i^2}{2}}}{\sqrt{2\pi}} dx_n$$

(Putting $x_n = \sqrt{|X|^2 - S}$, $\frac{\partial x_n}{\partial |X|} = \frac{|X|}{\sqrt{|X|^2-S}} = \frac{\sqrt{x_N^2+S}}{x_n}$)

$$\begin{aligned}
&= \int_0^{\frac{d_1}{2}} da_1 \cdots \int_0^{\frac{d_n}{2}} da_n \int_{-\infty}^{\infty} dx_1 \cdots \\
&\qquad \cdots \int_{-\infty}^{\infty} dx_{n-1} \prod_{i=1}^{n-1} \frac{e^{-\frac{x_i^2}{2}}}{\sqrt{2\pi}} \int_{\sqrt{M^2-S}}^{\infty} \frac{x_n}{\sqrt{x_n^2+S}} \cdot \frac{e^{-\frac{x_n^2}{2}}}{\sqrt{2\pi}} dx_n\\
&= \int_0^{\frac{d_1}{2}} da_1 \cdots \int_0^{\frac{d_n}{2}} da_n \int_{-\infty}^{\infty} dx_1 \cdots \int_{-\infty}^{\infty} \frac{e^{\frac{S}{2}} \Gamma\left(\frac{1}{2}, \frac{M^2}{2}\right)}{2\sqrt{\pi}} \prod_{i=1}^{n-1} \frac{e^{-\frac{x_i^2}{2}}}{\sqrt{2\pi}} dx_{n-1}
\end{aligned}$$

The function is rather complex to integrate and is left as an exercise for the reader.

Case 2: Negative x_n

In this case, $x_n = -\sqrt{|X|^2 - S}$, meaning AB may intersect the hyperplanes. Precisely, it will do so when

$$\begin{aligned}
\frac{-a_n \cdot |X|}{l} &\geq x_n = -\sqrt{|X|^2 - S} \\
\Leftrightarrow \frac{a_n \cdot |X|}{l} &\leq \sqrt{|X|^2 - S} \\
\Leftrightarrow \frac{a_n^2 \cdot |X|^2}{l^2} &\leq |X|^2 - S \\
\Leftrightarrow (a_n^2 - l^2)|X|^2 &\leq -Sl^2 \\
\Leftrightarrow |X| &\geq \frac{Sl^2}{l^2 - a_n^2}.
\end{aligned}$$

Therefore,

$$\begin{aligned}
&P(\text{not intersect when } x_n \text{ is negative}) \\
&= \int_0^{\frac{d_1}{2}} da_1 \cdots \int_0^{\frac{d_n}{2}} da_n \int_{-\infty}^{\infty} dx_1 \cdots \\
&\qquad \cdots \int_{-\infty}^{\infty} dx_{n-1} \int_{\max\left(M, \frac{Sl^2}{l^2 - a_n^2}\right)}^{\infty} \frac{e^{-\frac{|X|^2 - S}{2}}}{\sqrt{2\pi}} \prod_{i=1}^{n-1} \frac{e^{-\frac{x_i^2}{2}}}{\sqrt{2\pi}} d|X| \\
&= \int_0^{\frac{d_1}{2}} da_1 \cdots \int_0^{\frac{d_n}{2}} da_n \int_{-\infty}^{\infty} dx_1 \cdots \\
&\qquad \cdots \int_{-\infty}^{\infty} dx_{n-1} \int_{\sqrt{\max\left(M, \frac{Sl^2}{l^2 - a_n^2}\right)^2 - S}}^{\infty} \frac{x_n}{\sqrt{x_n^2 + S}} \prod_{i=1}^{n} \frac{e^{-\frac{x_i^2}{2}}}{\sqrt{2\pi}} dx_n.
\end{aligned}$$

(Putting $x_n = \sqrt{|X|^2 - S}$)

Putting all together, we get

Theorem 5.1. *The probability that a needle of length l intersects hyperplanes on grid space $d_1, d_2, \cdots, d_n$ is*

$$\int_0^{\frac{d_1}{2}} da_1 \cdots \int_0^{\frac{d_n}{2}} da_n \int_{-\infty}^{\infty} dx_1 \cdots$$

$$\cdots \int_{-\infty}^{\infty} dx_{n-1} \int_{\max\left(x_1^2+x_2^2+\cdots+x_{n-1}^2,\ \max\limits_{1\le i<n} \frac{l\cdot x_i}{-a_i}\right)}^{\infty} \frac{e^{-\frac{|X|^2-S}{2}}}{\sqrt{2\pi}} \prod_{i=1}^{n-1} \frac{e^{-\frac{x_i^2}{2}}}{\sqrt{2\pi}} d|X|$$

$$+ \int_0^{\frac{d_1}{2}} da_1 \cdots \int_0^{\frac{d_n}{2}} da_n \int_{-\infty}^{\infty} dx_1 \cdots$$

$$\cdots \int_{-\infty}^{\infty} dx_{n-1} \int_{\max\left(x_1^2+x_2^2+\cdots+x_{n-1}^2,\ \max\limits_{1\le i<n} \frac{l\cdot x_i}{-a_i},\ \frac{Sl^2}{l^2-a_n^2}\right)}^{\infty} \frac{e^{-\frac{|X|^2-S}{2}}}{\sqrt{2\pi}} \prod_{i=1}^{n} \frac{e^{-\frac{x_i^2}{2}}}{\sqrt{2\pi}} dx_n$$

6. Conclusion

6.1. Summary.

We proposed two generalizations and two variations of the Buffon-Laplace Needle Problem. The generalizations are dropping regular polygons or arbitrary polygons on a 2-dimensional space instead of a needle, and variations are dropping right regular polygon prism in 3-dimensional space and dropping needles in n-dimensional space. In each problem, we have given a formula for the answer. We have also given an efficient algorithm in the arbitrary polygon problem.

6.2. Possible Directions for Future Research.

We may extend the 2D arbitrary polygon problem to 3D. The method of finding the algorithm for the 2D arbitrary polygon problem may be used in the problem. We may also give an efficient algorithm for finding the probability that a 3D arbitrary polyhedron being dropped randomly intersects with one or more grid planes of the cuboidal grid in 3D space.

We may also change the grid line into different types, such as concentric circles, or regular polygon gridlines. For regular polygon gridlines, the algorithm given in the 2D arbitrary polygon problem may be modified and used in that problem.

Our investigation does not have any real-life applications. We may consider the *Happy Rainbow* Game that is easily found in theme parks and arcades. It is similar to what we are investigating on. We may apply some of our methods use in this paper to solve this problem.

7. Appendix

7.1. A Monte-Carlo Simulation of the regular polygon problem written in C++.

```
#include <algorithm>
#include <cmath>
#include <cstdio>
```

```
#include <cstdlib>
#include <random>
#include <chrono>

double constexpr PI = 3.14159265;

int main(int argc, char **argv)
{
    if (argc < 2 || argc > 6)
    {
        printf("Usage: %s number-of-sides [radius] [number-of-trials]
        [grid-height][grid-width]\n", argv[0]);
        return 1;
    }

    int number_of_sides = std::atoi(argv[1]),
        number_of_trials = argc >= 4 ? std::atoi(argv[3]) : 1000;
    double radius = argc >= 3 ? std::atof(argv[2]) : 1,
        grid_height = argc >= 5 ? std::atof(argv[4]) : 1,
        grid_width = argc >= 6 ? std::atof(argv[5]) : 1;

    std::mt19937_64 generator(std::chrono::system_clock::now().
    time_since_epoch().count());
    std::uniform_real_distribution<double>
        x_distribution(0.0, grid_width),
        y_distribution(0.0, grid_height),
        theta_distribution(0.0, 2 * PI / number_of_sides);

    int number_of_intersections = 0;
    for (int i = 0; i < number_of_trials; ++i)
    {

        double
            x = x_distribution(generator),
            y = y_distribution(generator),
            theta = theta_distribution(generator);
            double min_x = 1e9, min_y = 1e9, max_x = -1e9, max_y = -1e9;

        for (int j = 0; j < number_of_sides; ++j)
            {
                double current_angle = 2 * PI * j / number_of_sides + theta;
                min_x = std::min(min_x, radius * std::cos(current_angle));
                min_y = std::min(min_y, radius * std::sin(current_angle));
                max_x = std::max(max_x, radius * std::cos(current_angle));
                max_y = std::max(max_y, radius * std::sin(current_angle));
            }

            if (x + min_x <= 0 || y + min_y <= 0 || x + max_x >= grid_width
            || y + max_y >= grid_height)
                 ++number_of_intersections;
```

```
    }

    printf("Simulation probability:  %.10f (%d/%d)\n",
        (double)number_of_intersections / number_of_trials,
        number_of_intersections,
        number_of_trials);
}
```

Also included in the submission of this paper.

References

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[Buf33] G Buffon. Editor's note concerning a lecture given 1733 by mr. le clerc de buffon. *de l'Acad. Roy. des. Sciences*, pages 43–45, 1733.

[GS] Convex hull — set 2 (graham scan). https://www.geeksforgeeks.org/convex-hull-set-2-graham-scan/.

[Lap20] P.S. Laplace. *Théorie analytique des probabilités.* Paris: Veuve Courcier, 1820.

[PD] Probability density function. https://en.wikipedia.org/wiki/Probability_density_function.

[RM] Rotation matrix. https://en.wikipedia.org/wiki/Rotation_matrix.

REVIEWERS' COMMENTS

This paper studied a classic problem in probability theory – the Buffon-Laplace Needle Problem – which asks for the probability that a needle of length l will land on at least one line, given a floor with a rectangular grid of length a and width b. The authors studied various generalizations of the problem, such as replacing the needle by a regular polygon, and also to higher-dimensional grids. Reviewers think that these generalizations are very natural and meaningful, and the results are non-trivial as the proof involve the computation of complicated multiple integrals. Reviewers find that some results made use of assumptions on the size of the polygons and the grids, and wish the authors would have stated them clearly in the paper.