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INVESTIGATION ON THE ERDŐS-STRAUS CONJECTURE

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Abstract. In this paper, we are going to investigate the ***Erdős-Straus Conjecture*** : For any positive $n \geq 2$, there exists positive integers k, k_1, k_2 such that

$$\frac{4}{n} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}$$

Firstly, we will solve a simpler form $\frac{3}{n} = \frac{1}{x} + \frac{1}{y}$ as a starting point. Next we will investigate the Erdős-Straus Conjecture in the following dimensions: the related geometric representation of the Erdős-Straus Conjecture, the properties of solutions of the Erdős-Straus Conjecture, further investigation of some paper of the Erdős-Straus Conjecture, existence of special forms of solutions of the Erdős-Straus Conjecture, and the investigation of the Erdős-Straus Conjecture in algebraic dimension. The aim of this report is to find evidence that shows the ***Erdős-Straus Conjecture*** is true. If evidence is not strong enough, we still hope that this report can make an improvement to the researched result at present.

1. Notation.

1.1 p is a prime number.
1.2 (p, q) represents the greatest common divisor of p, q.
1.3 p, q are called relatively prime when $(p, q) = 1$.
1.4 $\mathbb{N}$ = the set of all natural numbers.
1.5 $\mathbb{Z}$ = the set of all integers.
1.6 $\mathbb{Q}$ = the set of all rational numbers.
1.7 $\mathbb{R}$ = the set of all real numbers.
1.8 k, k_1, k_2 are the solutions of the Erdős-Straus Conjecture if there exists a prime number such that

$$\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}.$$

1.9 Two integers a, b are said to be congruent to modular m if the remainders of a, b divided by m are the same. We notify it by $a \equiv b \pmod{m}$.

2. Preliminary Knowledge:

2.1 Perpendicular Distance Formula

Given $L : Ax + By + C = 0$. Then the perpendicular distance from a point (p, q) to the line L is

$$d = \frac{|Ax + By + C|}{\sqrt{A^2 + B^2}}.$$

2.2 Lens formula

If the distances from the object to the lens and from the lens to the image are u and v respectively, for a lens of negligible thickness, in air, the distances are related by the **thin lens formula**:

$$\frac{1}{f} = \frac{1}{u} + \frac{1}{v}.$$

2.3 Chinese remainder theorem

For any given sequence of integers $a_1, \ldots, a_k$, there exists an integer x solving the following system of simultaneous congruences.

$$\begin{cases} x \equiv a_1 \pmod{n_1} \\ \ldots\ldots \\ x \equiv a_k \pmod{n_k} \end{cases} \quad \text{where } (n_i, n_j) = 1, i \neq j \leq k.$$

The solution of x is given by

$$x \equiv a_1 t_1 + a_2 t_2 + \cdots + a_k t_k \pmod{lcm(n_1, n_2, \ldots, n_k)}$$

where A_m satisfy

$$t_m = \frac{lcm(n_1, n_2, \ldots, n_k)}{n_m} \times A_m \equiv 1 \pmod{n_m}$$

We will use it in Excel file for calculation.

3. Introduction

Erdős-Straus Conjecture:

For any positive $n \geq 2$, there exists positive integers k, k_1, k_2 such that

$$\frac{4}{n} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}.$$

Egypt fractions is a topic researched for a long time, but until now we still can't understand all of its properties. The Erdős-Stratus Conjecture is one of the famous open problems in it. It is known that this conjecture is true when $p < 10^{14}$ [1], but the existence of its solutions for all prime p remains a mystery. Why we only need to consider prime number p will be explained later. Therefore this report aims to find evidence that support this conjecture. In addition to algebraic dimensions, we will provide a geometric model of $\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}$ and use it as a new dimension discovering this conjecture.

4. The form $\frac{3}{n} = \frac{1}{x} + \frac{1}{y}$ and its related geometric representation

It is too complicated for us to deal with the Erdős-Stratus conjecture directly. So we start from finding the relationship between one particular fraction (i.e. $\frac{3}{n}$) and the sum of 2 unit fractions, and see that can it help us to understand about the addition of unit fractions.

Theorem 1. *Given that $n \in \mathbb{N}$. Then, all positive integral solutions (x, y) of $\frac{1}{n} = \frac{1}{x} + \frac{1}{y}$ are given by $x = n + s$ and $y = n + \frac{n^2}{s}$, i.e.*

$$\frac{1}{n} = \frac{1}{n+s} + \frac{1}{n + \frac{n^2}{s}}$$

where $s \in \mathbb{N}$ and $\frac{n^2}{s} \in \mathbb{N}$.

First proof:

$$\frac{1}{n} = \frac{1}{a} + \frac{1}{b}$$
$$\frac{a-n}{an} = \frac{1}{b}$$
$$b = \frac{an}{a-n} \tag{1}$$

From (1), we have $a > n$. Since $b \in \mathbb{N}$, then $(a - n) \mid an$, $an = k(a - n)$ where k is a positive integer.

$$a = \frac{kn}{k-n} \tag{2}$$

Let $s = k - n$ and then $s \in \mathbb{N}$ for $k > n$. By (2), we have

$$a = \frac{n(s+n)}{s}$$
$$a = n + \frac{n^2}{s} \tag{3}$$

From (3), we have $\frac{n^2}{s} \in \mathbb{N}$ for $a > n$.

Put (3) into (1):

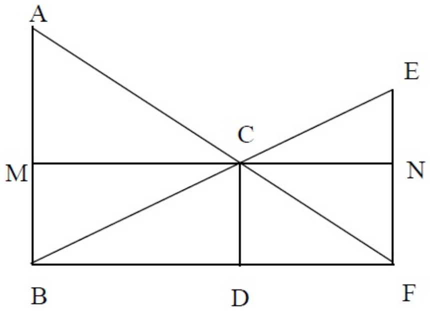
$$b = \frac{n\left(n + \dfrac{n^2}{s}\right)}{\dfrac{n^2}{s}} = s + n$$

Hence

$$\frac{1}{n} = \frac{1}{n+s} + \frac{1}{n + \dfrac{n^2}{s}}$$

where $s \in \mathbb{N}$ and $\dfrac{n^2}{s} \in \mathbb{N}$. □

Second proof: The second proof shows the geometric idea of the Theorem by using the idea of lens formula $\mathbf{\frac{1}{f} = \frac{1}{u} + \frac{1}{v}}$.



Consider the above figure where $AB//CD//EF$, $CM \perp AB$ and $CN \perp EF$.

We will first prove the relationship $\dfrac{1}{CD} = \dfrac{1}{AB} + \dfrac{1}{EF}$.

Proof. We know that $\triangle CDF \sim \triangle FAB$ and $\triangle BCD \sim \triangle BEF$

Then

$$\frac{CD}{EF} = \frac{BD}{BF} \tag{1}$$

$$\text{and} \quad \frac{CD}{AB} = \frac{DF}{FB} \tag{2}$$

(1) + (2):

$$\frac{CD}{AB} + \frac{CD}{EF} = \frac{BD + DF}{BF}$$

$$\therefore \quad \frac{1}{CD} = \frac{1}{AB} + \frac{1}{EF}\,\frac{AM}{CD} = \frac{s}{n} = \frac{BD}{DF} = \frac{CD}{EN}$$

Therefore, $\dfrac{1}{CD} = \dfrac{1}{AB} = \dfrac{1}{EF}$ □

By letting $CD = n$ and $AM = s$, we have $AB = n + s$.

Also, $CD = \frac{n^2}{s}$, we have $\frac{1}{n} = \frac{1}{n+s} + \frac{1}{n + \frac{n^2}{s}}$. □

Third Proof: For every $n \in N$, a relationship $\frac{1}{n} = \frac{1}{a} + \frac{1}{b}$ must exist where $a, b > n$ and $a, b \in \mathbb{N}$.

Hence we let $a = n + s$ and $b = n + r$. Then

$$\frac{1}{n+r} = \frac{1}{n} - \frac{1}{n+s} = \frac{1}{\frac{n(n+s)}{s}} \implies n + r = n + \frac{n^2}{s} \implies r = \frac{n^2}{s}.$$

Hence we can discover that for every $n \in \mathbb{N}, s \mid n^2 \in \mathbb{N}$.

For $\frac{1}{n} = \frac{1}{a} + \frac{1}{b}$, we have $\frac{1}{n} = \frac{1}{n+s} + \frac{1}{n + \frac{n^2}{s}}$ where $s \in \mathbb{N}$ and $\frac{n^2}{s} \in \mathbb{N}$. □

We have proved that we can find a way to represent $\frac{1}{n}$ as sum of two unit fractions, but for $\frac{m}{n}$, where $1 < m \in \mathbb{N}$, we still do not have a conclusion. Here we are going to find out the result of the case when $m = 3$.

Theorem 2. $\frac{3}{n} = \frac{1}{x} + \frac{1}{y}$ *exists for some* $x, y \in \mathbb{N}$ *if and only if*

(i) $n = 3k$ *or (ii)* $n = 3k + 2$ *or (iii)* $n = 3k + 1$

where there exists a positive integer $f \mid n$ *such that* $f \equiv 2 \pmod 3$.

Proof. The "if" part:

(i) When $n = 3k$, then $\frac{3}{n} = \frac{1}{k}$. By Theorem 1, there exists x, y to have

$$\frac{1}{k} = \frac{1}{x} + \frac{1}{y}.$$

Hence, $\frac{3}{n} = \frac{1}{x} + \frac{1}{y}$.

(ii) When $n = 3k + 2$, let $x = k + 1$ and $y = (k + 1)(3k + 2)$. Then,

$$\frac{1}{x} + \frac{1}{y} = \frac{1}{k+1} + \frac{1}{(k+1)(3k+1)} = \frac{3}{3k+2} = \frac{3}{n}$$

(iii) When $n = 3k+1$, if there exists where $f \mid (3k+1)$ such that $f \equiv 2 \pmod 3$, then we can construct $x = 3k+1+f \in \mathbb{N}$ and $y = 3k+1+\dfrac{(3k+1)^2}{f} \in \mathbb{N}$. By direct checking:

$$\frac{1}{x}+\frac{1}{y} = \frac{1}{3k+1+f} + \frac{1}{3k+1+\dfrac{(3k+1)^2}{f}} = \frac{1}{3k+1}$$

Since $3k+1+f \in \mathbb{N}$ and $3k+1+\dfrac{(3k+1)^2}{f} \in \mathbb{N}$,

$$\therefore \frac{3}{n} = \frac{1}{x}+\frac{1}{y}$$

The "only if" part:

Consider the excluded case, i.e Case (iv) :

$n = 3k+1$ where there exists **no** positive integer $f \mid n$ such that $f \equiv 2 \pmod 3$.

We assume solutions of $\dfrac{3}{n} = \dfrac{1}{x}+\dfrac{1}{y}$ exist in this case. Then from Theorem 1, all solutions of $\dfrac{3}{n} = \dfrac{1}{x}+\dfrac{1}{y}$ are given by

$$x = \frac{1}{\left(\dfrac{3k+1+f}{3}\right)} \quad \text{and} \quad y = \frac{1}{\left(\dfrac{3k+1+\dfrac{(3k+1)^2}{f}}{3}\right)}$$

Since $\dfrac{3k+1+f}{3}$ and $\dfrac{3k+1+\dfrac{(3k+1)^2}{f}}{3} \in \mathbb{N}$,

$$\therefore f \equiv 2 \pmod 3$$

But there exists **no** positive integer $f \mid n$ such that $f \equiv 2 \pmod 3$. Contradiction!

Therefore by rejecting the excluded case, $\dfrac{3}{n} = \dfrac{1}{x}+\dfrac{1}{y}$ exists for some $x, y \in \mathbb{N}$ if and only if (i) $n = 3k$ or (ii) $n = 3k+2$ or (iii) $n = 3k+1$ where there exists a positive integer $f \mid n$ such that $f \equiv 2 \pmod 3$. □

5. The Erdős-Straus Conjecture and its related geometric representation

Original Erdős-Straus Conjecture

For any positive $n \geq 2$, there exists positive integers k, k_1, k_2 such that
$$\frac{4}{n} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}.$$
However, we only need to consider n as a prime p. When $n = pq$ where p is a prime and $\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}$, then n also satisfies the Erdős-Straus Conjecture because
$$\frac{4}{n} = \frac{4}{pq} = \left(\frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}\right)\left(\frac{1}{q}\right) = \frac{1}{qk} + \frac{1}{qk_1} + \frac{1}{qk_2}$$
Hence, we only need to consider the amended Conjecture:

Erdős Straus Conjecture (amended):

For any positive prime $p \geq 2$, there exists positive integers k, k_1, k_2 such that
$$\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}.$$

Without notification, we will consider this amended Conjecture in the remaining part of the paper.

Also, without loss of generality: We let $\mathbf{k \leq k_1 \leq k_2}$ and p is prime.

Geometric Consideration of Erdős Straus Conjecture:

Figure 1 shows a $\triangle EBF$ with in-circle with centre H and its radius p and three ex-circles with centers K, M and N and their corresponding radii be $4k, 4k_1, 4k_2$ respectively. Also, O is the origin of the coordinate plane with these in-circles and 3 ex-circles touching the axes as shown below. In addition, D, A, C be the points of contact of circles with centre M, H, K and x-axis respectively.

Theorem 3. *Any triangle with r be the radius of the in-circle and x, y, z be the radii of 3 ex-circles respectively has:*
$$\frac{1}{r} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}.$$

Proof. We can refer to Figure 1.

Let $FB = a, EB = c$ and $FE = b$ and x, y and z be the radii of the circles with the centers K, N and M respectively.

Consider $FKBE$,
$$\text{the area of } \triangle FEB = \text{area of } FKBE - \text{area of } \triangle FKB \quad (1)$$

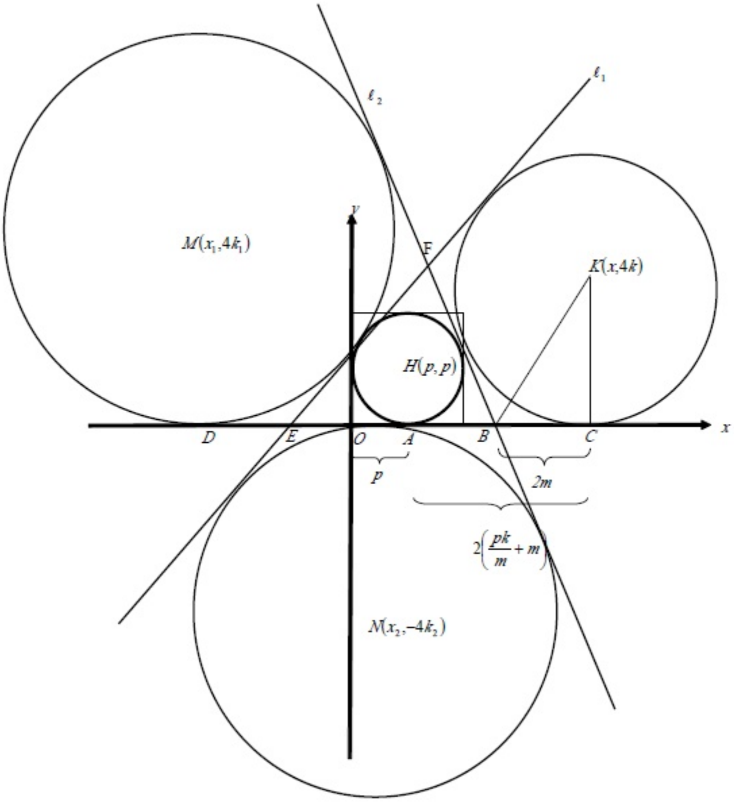


Figure 1

By tangent properties, area of $FKBE = \frac{1}{2}bx + \frac{1}{2}cx$, area of $\triangle FKB = \frac{1}{2}ax$.

Also by the properties of incentre of triangle, the area of $\triangle FEB = rs$ where $s = \dfrac{a+b+c}{2}$.

Then (1) becomes

$$(b+c-a)x = 2rs \tag{2}$$

Similarly, by considering $NEFB$ and $MFBE$, we can set up the other two equations:

$$(a+b-c)y = 2rs \tag{3}$$

$$(c+a-b)x = 2rs \tag{4}$$

(2)+(3)+(4),

$$2a + 2b + 2c - a - b - c = \frac{2rs}{x} + \frac{2rs}{y} + \frac{2rs}{z}$$
$$\implies \frac{1}{r} = \frac{1}{x} + \frac{1}{y} + \frac{1}{z}$$

□

Theorem 4. *Refer to Figure 1. Let* $H(p,p), K(x,4k), M(x_1,4k_1), N(x_2,-4k_2)$ *be the centres of the middle, right, left and the bottom circles respectively with* $k \le k_1 \le k_2$*. We have:*

$$\begin{cases} k = k \\ k_1 = \left(\dfrac{m^2+pk}{4k-p}\right) \\ k_2 = \dfrac{pk}{m^2}\left(\dfrac{m^2+pk}{4k-p}\right) \end{cases}$$

where $BC = 2m$*.*

Proof. By knowing that $k \le k_1 \le k_2$, we have the radii $4k \le 4k_1 \le 4k_2$ and the circle with centre N is the largest ex-circle and the circle with centre K is the smallest ex-circle. From the proof of Theorem 3, $EB = c, EF = b$ and $FB = a$.

By (2), (3), (4) from the previous proof, we have

$$4k_2 = \frac{2rs}{(2s-2c)} = \frac{rs}{s-c},$$
$$4k_1 = \frac{rs}{s-b},$$
$$4k = \frac{rs}{s-a}.$$

Then $4k \le 4k_1 \le 4k_2 \implies c \ge b \ge a$.

We have $EB \ge EF \ge FB$ and $\angle EFB$ is the greatest angle.

Then $\angle FBE$ and $\angle FEB$ are acute because $\angle EFB$ is the greatest angle.

Let θ be $\angle BKC$. Then $\angle ABH = \angle BKC = \theta$.

Let $BC = 2m$.

In $\triangle BKC$,

$$\tan\theta = \frac{2m}{4k} = \frac{m}{2k}.$$

In $\triangle ABH$,

$$AB = \frac{p}{\tan\theta} = \frac{p}{\frac{m}{2k}} = \frac{2pk}{m}.$$

Hence, $AC = \dfrac{2pk}{m} + 2m = 2\left(\dfrac{pk}{m} + m\right)$.

For $\triangle AEH \sim \triangle CEK$,

$$\frac{EO+p}{EO+p+AC} = \frac{p}{4k}$$
$$EO = \frac{p^2 + pAC - 4kp}{4k-p}$$
$$EO = \frac{p^2 + p(2)\left(\dfrac{pk}{m} + m\right) - 4kp}{4k-p}$$

Also,

$$OB = p + AC - 2m$$
$$OB = p + 2\left(\frac{pk}{m} + m\right) - 2m = p + \frac{2pk}{m};$$
$$EB = EO + OB$$
$$EB = \frac{p^2 + \dfrac{2p^2k}{m} + 2pm - 4kp}{4k-p} + p + \frac{2pk}{m}$$
$$EB = \frac{p^2 + \dfrac{2p^2k}{m} + 2pm - 4kp + 4kp + \dfrac{8pk^2}{m} - p^2 - \dfrac{2p^2k}{m}}{4k-p}$$
$$EB = \frac{2pm + \dfrac{8pk^2}{m}}{4k-p} = \frac{2p}{m}\left(\frac{m^2+4k^2}{4k-p}\right).$$

Now, let $l_1 : m_1x - y + c_1 = 0$ and $l_2 : m_2x - y + c_2 = 0$.

Let $N(x_2, -4k_2) = (a_2, -r_2)$.

If $(a_2, -r_2)$ is below both l_1, l_2, then $m_1a_2 - (-r_2) + c_1 > 0$ and $m_2a_2 - (-r_2) + c_2 > 0$. Hence, by perpendicular distance formula, we have

$$r_2 = \frac{m_1a_2 - (-r_2) + c_1}{\sqrt{m_1^2+1}} = \frac{m_2a_2 - (-r_2) + c_2}{\sqrt{m_2^2+1}}.$$

By making a_2 as the subject, we have

$$\frac{r_2(\sqrt{1+m_2^2} - 1) - c_2}{m_2} = \frac{r_2(\sqrt{1+m_1^2} - 1) - c_1}{m_1}$$

$$r_2 = \frac{m_1c_2 - m_2c_1}{m_1(\sqrt{1+m_2^2} - 1) - m_2(\sqrt{1+m_1^2} - 1)}.$$

For $c_1 = EO\tan\theta_1$ and $c_2 = -OB\tan\theta_2$ where θ_1, θ_2 are the inclinations of l_1 and l_2 respectively.

$$r_2 = \frac{-(OB+EO)\tan\theta_1\tan\theta_2}{\tan\theta_1(\sqrt{1+\tan^2\theta_2}-1)-\tan\theta_2(\sqrt{1+\tan^2\theta_1}-1)}$$
$$r_2 = \frac{-EB\times\sin\theta_1\sin\theta_2}{-\sin\theta_1-\sin\theta_1\cos\theta_2-\sin\theta_2+\sin\theta_2\cos\theta_1}$$

for $\sqrt{1+\tan^2\theta_1} = \dfrac{1}{\cos\theta_1}$ and $\sqrt{1+\tan^2\theta_2} = \dfrac{-1}{\cos\theta_2}$.

For $\sin\theta_1 = \dfrac{2t}{1+t_1^2}$, $\sin\theta_2 = \dfrac{2t_2}{1+t_2^2}$, $\cos\theta_1 = \dfrac{1-t_1^2}{1+t_1^2}$ and $\cos\theta_2 = \dfrac{1-t_2^2}{1+t_2^2}$ and $r_2 = 4k_2$ where $t_1 = \tan\dfrac{\theta_1}{2}$ and $t_2 = \tan\dfrac{\theta_2}{2}$.

Then, we have

$$4k_2 = \frac{-EB\times 4t_1t_2}{-2t_1-2t_1t_2^2-2t_1+2t_1t_2^2-2t_2-2t_2t_1^2+2t_2-2t_2t_1^2}$$
$$4k_2 = \frac{-EB\times 4t_1t_2}{-4t_1-4t_2t_1}$$
$$4k_2 = \frac{EB\times t_2}{1+t_1t_2}.$$

Let $M(x_1, 4k_1) = (a_1, r_1)$.

If (a_1, r_1) is above l_1 and below l_2, then $m_1a_1 - r_1 + c_1 < 0$ and $m_2a_1 - r_1 + c_2 > 0$.

Hence, by perpendicular distance formula, we have

$$r_1 = -\left(\frac{m_1a_1 - r_1 + c_1}{\sqrt{m_1^2+1}}\right) = \frac{m_2a_1 - r_1 + c_2}{\sqrt{m_2^2+1}}.$$

By making a_2 as the subject, we have:

$$\frac{r_1(1-\sqrt{m_1^2+1})-c_1}{m_1} = \frac{r_1(\sqrt{m_2^2+1}+1)-c_2}{m_2},$$
$$r_2 = \frac{m_2c_1 - m_1c_2}{m_2(1-\sqrt{m_1^2+1})-m_1(1+\sqrt{m_2^2+1})}.$$

For $c_1 = EO\tan\theta_1$ and $c_2 = -OB\tan\theta_2$ where θ_1, θ_2 are the inclinations of l_1 and l_2 respectively.

$$r_1 = \frac{(OB+EO)\tan\theta_1\tan\theta_2}{\tan\theta_2(1-\sqrt{1+\tan^2\theta_1})-\tan\theta_1(\sqrt{1+\tan^2\theta_2}+1)},$$
$$r_1 = \frac{EB\times\sin\theta_1\sin\theta_2}{\sin\theta_2\cos\theta_1-\sin\theta_2-\sin\theta_1\cos\theta_2+\sin\theta_1}$$

for $\sqrt{1+\tan^2\theta_1} = \dfrac{1}{\cos\theta_1}$ and $\sqrt{1+\tan^2\theta_2} = \dfrac{-1}{\cos\theta_2}$.

For $\sin\theta_1 = \dfrac{2t}{1+t_1^2}$, $\sin\theta_2 = \dfrac{2t_2}{1+t_2^2}$, $\cos\theta_1 = \dfrac{1-t_1^2}{1+t_1^2}$ and $\cos\theta_2 = \dfrac{1-t_2^2}{1+t_2^2}$ and $r_1 = 4k_1$ where $t_1 = \tan\dfrac{\theta_1}{2}$ and $t_2 = \tan\dfrac{\theta_2}{2}$. Then, we have

$$\begin{aligned}
4k_1 &= \frac{EB \times 4t_1t_2}{(2t_2)(1-t_1^2) - 2t_2(1+t_1^2) - 2t_1(1-t_2^2) + 2t_1(1+t_2^2)},\\
4k_1 &= \frac{EB \times 4t_1t_2}{-4t_2t_1^2 + 4t_1t_2^2},\\
4k_1 &= \frac{EB}{t_2 - t_1},\\
k_2 &= \frac{EB}{4(t_2-t_1)}.
\end{aligned}$$

For

$$\begin{aligned}
t_1 &= \frac{4k-p}{4q} = \frac{(4k-p)m}{2(pk+m^2)},\\
t_1 &= \tan\frac{\theta_2}{2} = \frac{2k}{m}.
\end{aligned}$$

Also, we have previously,

$$EB = \frac{2p}{m}\left(\frac{m^2+4k^2}{4k-p}\right).$$

Therefore,

$$\begin{aligned}
k_2 &= \frac{EBt_2}{1+t_1t_2}\\
&= \frac{\dfrac{2p}{m}\left(\dfrac{m^2+4k^2}{4k-p}\right)\left(\dfrac{2k}{m}\right)}{4\left(1+\dfrac{(4k-p)m}{2(pk+m^2)}\cdot\left(\dfrac{2k}{m}\right)\right)}\\
&= \frac{\dfrac{p}{k}\left(\dfrac{m^2+4k^2}{4k-p}\right)}{4\left(\dfrac{pk+m^2+4k^2-pk}{pk+m^2}\right)}\\
&= \frac{\dfrac{4pk}{m^2}\left(\dfrac{m^2+4k^2}{4k-p}\right)}{4\left(\dfrac{m^2+4k^2}{pk+m^2}\right)}\\
&= \frac{pk}{m^2}\left(\frac{m^2+pk}{4k-p}\right).
\end{aligned}$$

$$\begin{aligned}
k_1 &= \frac{EB}{t_2 - t_1} \\
&= \frac{\dfrac{2p(m^2+pk)}{m(4k-p)}}{\dfrac{(4k-p)m}{2(pk+m^2)} - \left(\dfrac{2k}{m}\right)} \\
&= \frac{\dfrac{2p(m^2+pk)}{m(4k-p)}}{\left(\dfrac{4km^2 - pm^2 - 4k^2p - 4km^2}{2m(pk+m^2)}\right)} \\
&= \frac{pk+m^2}{4k-p}.
\end{aligned}$$

Hence, we have

$$\begin{cases} k = k \\ k_1 = \left(\dfrac{m^2+pk}{4k-p}\right) \\ k_2 = \dfrac{pk}{m^2}\left(\dfrac{m^2+pk}{4k-p}\right) \end{cases}$$

where $k \leq k_1 \leq k_2$. □

From the above result, we can relate this geometric result with the solutions of Erdős Straus Conjecture and this attempt may be better to explore the Conjecture because we can express k, k_1, k_2 separately by m^2 and k. Some results about the Conjecture can be obtained after the following Theorem 5.

Theorem 5. $\dfrac{4}{p} = \dfrac{1}{k} + \dfrac{1}{k_1} + \dfrac{1}{k_2}$ *where* k, k_1, k_2 *are positive integers if and only if*

$$\begin{cases} k = k \\ k_1 = \left(\dfrac{m^2+pk}{4k-p}\right) \\ k_2 = \dfrac{pk}{m^2}\left(\dfrac{m^2+pk}{4k-p}\right) \end{cases}$$

where k, k_1, k_2 *are positive integers and* $m^2 > 0$.

Proof. **Prove the "If part":**

Clearly,

$$\frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2} = \frac{1}{k} + \frac{4k-p}{m^2+pk} + \frac{m^2}{pk}\left(\frac{4k-p}{m^2+pk}\right) = \frac{4}{p}.$$

Prove the "Only if part":

If we have $\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}$ where k, k_1, k_2 are positive integers, then let $m^2 = \frac{pkk_1}{k_2}$.
Then, $m^2 > 0$. Since $\frac{k_1 k_2}{pk} = \frac{k_1 + k_2}{4k - p}$ by $\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}$, then

$$\begin{aligned}\frac{m^2 + pk}{4k - p} &= \frac{\frac{pkk_1}{k_2} + pk}{4k - p} \\ &= \frac{pk(k_1 + k_2)}{k_2(4k - p)} \\ &= \frac{pk}{k_2}\left(\frac{k_1 k_2}{pk}\right) \\ &= k_1.\end{aligned}$$

Also,

$$\begin{aligned}\frac{pk}{m^2}\left(\frac{m^2 + pk}{4k - p}\right) &= \frac{pk}{m^2} k_1 \\ &= \frac{pkk_1}{\left(\frac{pkk_1}{k_2}\right)} \\ &= k_2.\end{aligned}$$

□

Theorem 6. *If*

$$\begin{cases} k = k \\ k_1 = \left(\dfrac{m^2 + pk}{4k - p}\right) \\ k_2 = \dfrac{pk}{m^2}\left(\dfrac{m^2 + pk}{4k - p}\right) \end{cases}$$

where k, k_1, k_2 are positive integers, then $m^2 \in \mathbb{N}$ and $m^2 \mid p^2 k^2$.

Proof. $k_1 = \frac{m^2 + pk}{4k - p} \implies m^2 = k_1(4k - p) - pk$, then m^2 is an integer.

k_1, k, p are all positive integers and $m^2 = \frac{pkk_1}{k_2}$, hence, $m^2 \in \mathbb{N}$.

Also, $k_2 = \frac{pk}{m^2}\left(\frac{m^2 + pk}{4k - p}\right) = \frac{pk + \frac{(pk)^2}{m^2}}{4k - p}$ and k_2 is a positive integer.

Then, $m^2 \mid (pk)^2$. □

6. Properties of the Solutions of the Erdős Straus Conjecture:

6.1.

From a Japanese website

(www.asahi-net.or.jp/∼kc2h-msm/mathland/math01/erdstr00.htm),

we quote a list of the solutions of the Erdős Straus Conjecture when p is small:

4/p = 1/a + 1/b + 1/c

p	a	b	c
2	1	2	2
3	1	4	12
		6	6
	2	2	3
5	2	4	20
		5	10
7	2	15	210
		16	112
		18	63
		21	42
		28	28
	3	6	14
	4	4	14
11	3	34	1122
		36	396
		42	154
		44	132
		66	66
	4	9	396
		11	44
		12	33
	6	6	33
13	4	18	468
		20	130
		26	52
	5	10	130
17	5	30	510
		34	170
	6	15	510
		17	102
19	5	96	9120
		100	1900
		114	570
		120	456
		190	190
	6	23	2622
		24	456
		30	95
		38	57
	8	12	456
	10	10	95
23	6	139	19182
		140	9660
		141	6486
		142	4899
		144	3312
		147	2254
		150	1725
		156	1196
		161	966
		174	667
		184	552
		207	414
		230	345
		276	276
	7	42	138
	8	23	184
		24	138
	9	16	3312
		18	138

From the above list of the solutions, although p is small, we still could observe some patterns and obtain some corresponding Theorems as follows:

Theorem 7. *$k < p$ and then k is not divisible by p.*

Proof. Assume the contrary that $k \geq p$. For $k \leq k_1 \leq k_2$, then $p \leq k \leq k_1 \leq k_2$.

Since $\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}$, then

$$\begin{aligned} & \frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2} \leq \frac{1}{p} + \frac{1}{p} + \frac{1}{p} \\ \Longrightarrow \quad & \frac{4}{p} \leq \frac{3}{p} \\ \Longrightarrow \quad & 4 \leq 3. \end{aligned}$$

Contradiction!

Since k is positive and $0 < k < p$, k is not divisible by p. □

Theorem 8. *k_2 is divisible by p that is $p \mid k_2$.*

Proof. We have $k = k \leq k_1 = \dfrac{m^2 + pk}{4k - p} \leq k_2 = \dfrac{pk}{m^2}\left(\dfrac{m^2 + pk}{4k - p}\right)$. From Theorem 7, we have k is not divisible by p.

Case 1: If m^2 is not divisible by p, then $(p, m^2 + pk) = 1$. We have k_1 is also not divisible by p and k_2 is divisible by p.

Case 2: If m^2 is divisible by p, then k is also divisible by p because if $(4k - p)$ has no p as a factor, we have $\dfrac{m^2 + pk}{4k - p}$ has p as a factor k_1 that is divisible by p, otherwise, $(4k - p)$ is divisible by p, we have $p = 2$, then by $\frac{4}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{1} \implies k_1$is divisible by p.

Case 2(a): m^2 is not divisible by p^2

Then $k_2 = \dfrac{pk}{pm'^2}(k_1)$ where we let $m^2 = pm'^2$ and m'^2 is not divisible by p.

We have $k_2 = \dfrac{k}{m'^2}(k_1)$. For k_1 is also divisible by p, then k_2 is also divisible by p.

Case 2(b): m^2 is divisible by p^2.

Case 2(b)(i): If $k_1 = k_2$, then $pk = m^2$. Since k is not divisible by p then m^2 is not divisible by p^2. We have a contradiction.

Case 2(b)(ii): If $k_1 < k_2$, then

$$\begin{aligned}
&\Longrightarrow \frac{pk}{m^2} > 1 \quad \text{for} \quad k_2 = \frac{pk}{m^2}k_1 \\
&\Longrightarrow \frac{pk}{p^2m'^2} > 1 \quad \text{where we let } m^2 = p^2m'^2 \\
&\Longrightarrow k > pm'^2 \\
&\Longrightarrow k > p
\end{aligned}$$

Hence, $p < k \leq k_1 \leq k_2$, and we have

$$\begin{aligned}
\frac{4}{p} &< \frac{1}{p} + \frac{1}{p} + \frac{1}{p}, \\
\frac{4}{p} &< \frac{3}{p}.
\end{aligned}$$

We have contradiction!!! □

Theorem 9. *None of k, k_1, k_2 can be divisible by p^2.*

Proof. Case A: $p = 2$.

By $\frac{4}{2} = \frac{1}{2} + \frac{1}{2} + \frac{1}{1} \implies k = 1,\ k_1 = 2,\ k_2 = 2$. Hence none of k, k_1, k_2 can be divisible by p^2.

Case B:

By Theorem 7, k cannot be divisible by p and then k is also not divisible by p^2.

Case B1: k_1 is divisible by p^2.

Then, $m^2 + pk$ must be divisible by p^2. We have $p \mid m^2$ but m^2 is not divisible by p^2 from the previous proof, $k_2 = \frac{pk}{m^2}\left(\frac{m^2 + pk}{4p - k}\right)$ is also divisible by p^2 for m^2 is not divisible by p^2 and $(2, 4k - p) = 1$.

Case B2: k_2 is divisible by p^2.

From the previous proof, $p \mid m^2$ but m^2 is not divisible by p^2. Then $m^2 + pk$ must be divisible by p^2 for $k_2 = \frac{pk}{m^2}\left(\frac{m^2 + pk}{4p - k}\right)$ and $(2, 4k - p) = 1$. Hence, k_1 is also divisible by p^2.

Hence if one of k_1, k_2 is divisible by p^2, by Case B1 or CaseB2, both are divisible by p^2. Now, we assume that they are divisible by p^2. We can let $k_1 = p^2a$ and $k_2 = p^2b$. Then,

$$
\begin{aligned}
& & \frac{4}{n} &= \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2} \\
&\implies & \frac{4}{p} - \frac{1}{k} &= \frac{1}{p^2a} + \frac{1}{p^2b} \\
&\implies & \frac{4k-p}{pk} &< \frac{2}{p^2} \\
&\implies & \frac{4k-p}{k} &< \frac{2}{p} \\
&\implies & 4k-p &< \frac{2k}{p} \\
&\implies & k &< \frac{p}{4-\frac{2}{p}} \\
&\implies & k &< \frac{p}{4\left(1-\frac{1}{2p}\right)} \\
&\implies & k &< \frac{p}{4}\left(1+\frac{1}{2p}+\dots\right) \\
&\implies & k &< \frac{p}{4}+\frac{p}{4}\left(\frac{1}{2p}\right)\left(1+\frac{1}{2p}+\dots\right) \\
&\implies & k &< \frac{p}{4}+\frac{1}{8}\left(\frac{1}{1-\frac{1}{2p}}\right) \qquad (1)
\end{aligned}
$$

Also, $\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}$, then

$$\frac{4}{p} > \frac{1}{k} \qquad (2)$$

Combine (1) and (2), we have

$$
\begin{aligned}
\frac{p}{4} < \quad k \ &< \frac{p}{4}+\frac{1}{8}\left(\frac{1}{1-\frac{1}{2p}}\right) \\
p < 4k \ &< p+\frac{1}{2\left(1-\frac{1}{2p}\right)}
\end{aligned}
$$

As,

$$\begin{aligned} 1 &< p \\ p &< 2p-1 \\ \frac{p}{2p-1} &< 1 \\ \frac{1}{2\left(1-\frac{1}{2p}\right)} &< 1 \end{aligned}$$

Hence, $p < 4k < p+1$.

We have $4k$ is not an integer. We have contradiction!!! Therefore, both Case B1 and Case B2 are wrong. Hence, no k, k_1, k_2 can be divisible by p^2. □

6.2. The bounds of $\mathbf{k}, \mathbf{k_1}, \mathbf{k_2}$

Theorem 10. *The bounds of k, k_1, k_2 are*

$$\begin{cases} \frac{1}{4}p < k \ \leq \frac{3}{4}p \\ \ k \leq k_1 \leq \frac{3}{2}p^2 \\ k_1 \leq k_2 \leq \frac{9}{16}p^4 \end{cases}$$

Proof. The bounds for k, k_1, k_2:

If $\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}$, then

$$\begin{aligned} &\frac{1}{k} < \frac{4}{p} \text{ and } \frac{4}{p} \leq \frac{3}{k} \\ \implies \quad &\frac{p}{4} \leq k \leq \frac{3p}{4} \end{aligned}$$

Also,

$$\begin{aligned} k_1 &= \frac{m^2+pk}{4k-p} \\ &\leq \frac{pk+pk}{1} \text{ for } \frac{pk}{m^2} \geq 1 \text{ and } 4k-p \geq 1 \\ &= 2pk \\ &\leq 2p\left(\frac{3p}{4}\right) \\ &= \frac{3}{2}p^2 \end{aligned}$$

And,

$$\begin{aligned} k_2 &= \frac{pk}{m^2}\left(\frac{m^2+pk}{4k-p}\right) \\ &\leq \frac{pk}{1}(k_1) \text{ for } m^2 \geq 1 \\ &\leq pk\left(\frac{3p^2}{4}\right) \\ &\leq \frac{3}{4}p^3\left(\frac{3p}{4}\right) \\ &= \frac{9}{16}p^4 \end{aligned}$$

Hence,

$$\begin{cases} \frac{1}{4}p < k \ \leq \frac{3}{4}p \\ k \leq k_1 \leq \frac{3}{2}p^2 \\ k_1 \leq k_2 \leq \frac{9}{16}p^4 \end{cases}$$

□

6.3. Geometric Results related to Erdős-Straus Conjecture

Let k, k_1, k_2 be the solutions of the Erdős-Straus Conjecture.

Let $\triangle EBF$ be the triangle constructed in Figure 1.

Let the 3 sides of the triangle $\triangle EBF$ with the inscribed circle with radius p be a, b and c where $c \geq b \geq a$.

Let Δ be the area of the triangle. Then

$$\begin{aligned} \Delta &= \frac{1}{2}p(a+b+c) \text{ where } s = \frac{a+b+c}{2} \\ &= \frac{1}{2}p(2s) \\ &= ps. \end{aligned}$$

Also, we know from the proof of Theorem 4,

$$\begin{cases} 4k = \dfrac{ps}{s-a} \\ 4k_1 = \dfrac{ps}{s-b} \text{ where } c \geq b \geq a \\ 4k_2 = \dfrac{ps}{s-c}. \end{cases}$$

Hence,

$$\begin{aligned} 64kk_1k_2 &= \frac{(ps)^2}{(s-c)(s-b)(s-a)} \\ &= \frac{p^3s^3}{\frac{\Delta^2}{s}} \\ &= \frac{p^3s^3}{(ps)^2} \\ &= ps^2 \\ s^2 &= \frac{64kk_1k_2}{p} \end{aligned}$$

Since, by Theorem 8, $p \mid k_2$, then

$$\begin{aligned} s^2 &= \frac{64kk_1k_2}{p} \in \mathbb{N} \\ s^2 &= 64k\left(\frac{m^2+pk}{4k-p}\right)\frac{pk}{m^2}\left(\frac{m^2+pk}{4k-p}\right) \\ &= \frac{64k^2(m^2+pk)^2}{m^2(4k-p)^2} \\ s &= \frac{8k(m^2+pk)}{m(4k-p)} \\ s &= 8m\left(\frac{k_2}{p}\right) \end{aligned} \tag{1}$$

Since,

$$\begin{aligned} \Delta &= ps \\ \Delta &= p\left(\frac{8mk_2}{p}\right) = 8mk_2 \end{aligned} \tag{2}$$

In addition,

$$\begin{aligned}
s-a &= \frac{ps}{4k} \\
a &= \frac{s(4k-p)}{4k} \\
a &= \frac{8k(m^2+pk)(4k-p)}{m(4k-p)(4k)} \\
a &= \frac{2(m^2+pk)}{m} \\
a &= 2\left(m+\frac{pk}{m}\right) && (3)
\end{aligned}$$

And,

$$\begin{aligned}
s-b &= \frac{ps}{4k_1} \\
s-b &= \frac{\frac{8k(m^2+pk)}{m(4k-p)}p}{4\left(\frac{m^2+pk}{4k-p}\right)} \\
s-b &= 2\left(\frac{pk}{m}\right) \\
b &= s-\left(\frac{2pk}{m}\right) && (4) \\
b &= \frac{2k(4m^2+p^2)}{m(4k-p)} && (4')
\end{aligned}$$

And,

$$\begin{aligned}
s-c &= \frac{ps}{4k_2} \\
s-c &= \frac{\frac{8k(m^2+pk)}{m(4k-p)}p}{4\left(\frac{pk}{m^2}\right)\left(\frac{m^2+pk}{4k-p}\right)} \\
s-c &= 2m \\
c &= s-2m && (5) \\
c &= \frac{2p(4k^2+m^2)}{m(4k-p)} && (5')
\end{aligned}$$

With (1), (2), (3), (4) and (5), if $m \in \mathbb{N}$ and $m \mid 2pk \implies s, \Delta, a, b, c \in \mathbb{N}$. Hence, we have Theorem 11 as follows:

Theorem 11. *If k, k_1, k_2 are the solutions of the Erdős-Straus Conjecture that make $m \in \mathbb{N}$ and $m \mid 2pk$, then we can form a Herion triangle with sides a, b, c and area Δ and*

$$\begin{cases} s = 8m\left(\dfrac{k_2}{p}\right) \in \mathbb{N} \\ \Delta = 8mk_2 \in \mathbb{N} \\ c = 2\left(m + \dfrac{pk}{m}\right) \in \mathbb{N} \\ b = \dfrac{2k(4m^2+p^2)}{m(4k-p)} \in \mathbb{N} \\ a = \dfrac{2p(4k^2+m^2)}{m(4k-p)} \in \mathbb{N} \end{cases}$$

Proof. Proved before. □

From Theorem 11, if $p = 4k - 1$ where $k \in \mathbb{N}$, we can make $m = 1$, and we have a Herion triangle with

$$\begin{cases} \Delta = 8kp(1+pk) \in \mathbb{N} \\ c = 2p(4k^2+1) \in \mathbb{N} \\ b = 2k(4+p^2) \in \mathbb{N} \\ a = 2(1+pk) \in \mathbb{N} \end{cases}.$$

By reducing the size by a factor $\frac{1}{2}$ for lengths, we have a smaller Herion triangle with

$$\begin{cases} \Delta' = 2kp(1+pk) \in \mathbb{N} \\ c' = p(4k^2+1) \in \mathbb{N} \\ b' = k(4+p^2) \in \mathbb{N} \\ a' = (1+pk) \in \mathbb{N} \end{cases}$$

where Δ', a', b', c' are the corresponding area of the triangle and 3 sides respectively.

Theorem 12. *The Herion triangle formed in Theorem 11 cannot be a rational triangle.*

Proof. By Theorem 11, we have a Herion triangle with sides a, b, c and area Δ and

$$\begin{cases} a = 2\left(\dfrac{m^2+pk}{m}\right) \\ b = \dfrac{2k(4m^2+p^2)}{m(4k-p)} \\ c = \dfrac{2p(4k^2+m^2)}{m(4k-p)} \\ \Delta = \dfrac{8k(m^2+pk)p}{m(4k-p)} \end{cases}.$$

If the Herion triangle is a rational triangle, then

$$\begin{aligned} \frac{1}{2}ab &= \Delta \\ \frac{1}{2}\left(\frac{2k}{m}\right)\left(\frac{4m^2+p^2}{4k-p}\right) \times \frac{2(m^2+pk)}{m} &= \frac{8k}{m}\left(\frac{m^2+pk}{4k-p}\right)p \\ 4p &= \frac{4m^2+p^2}{m} \\ 0 &= 4m^2-4mp+p^2 \\ 0 &= (2m-p)^2 \\ p &= 2m \end{aligned}$$

$$p = 2 \text{ and } m = 1 \text{ for } p \text{ is prime.}$$

By Theorem 10, we have $\frac{1}{4}p < k \leq \frac{3}{4}p$, so k can only be 1.

Then, $k_2 = \dfrac{(1)^2+2(1)}{4(1)-2} \notin \mathbb{N}$.

Hence, the Herion triangle cannot be a rational triangle. □

Some Observations we have:

Consider $p = 4k-1$ where $k \in \mathbb{N}$ and make $m = 1$. Then, we obtained previously a Herion triangle $\triangle EBF$ and

$$\begin{cases} \Delta = 8kp(1+pk) \in \mathbb{N} \\ c = 2p(4k^2+1) \in \mathbb{N} \\ b = 2k(4+p^2) \in \mathbb{N} \\ a = 2(1+pk) \in \mathbb{N} \end{cases}.$$

Let h be the height corresponding to the largest base c of the Herion triangle $\triangle EBF$.

$$h = \frac{2\Delta}{c},$$
$$h = \frac{16kp(1+pk)}{2p(4k^2+1)} = \frac{8k(1+pk)}{4k^2+1}.$$

Although $\triangle EBF$ is not a rational triangle, we could obtain a rational triangle from $\triangle EBF$.

If we construct an altitude FG from F to EB of $\triangle EBF$. Let e be the base of the right-angled triangle $\triangle BGF$

$$\begin{aligned} e^2 &= a^2 - h^2 \\ &= (2(1+pk))^2 - \left(\frac{8k(1+pk)}{4k^2+1}\right)^2 \\ &= \frac{4(1+pk)^2(4k^2+1)^2 - 8^2k^2(1+pk)^2}{(4k^2+1)^2} \\ &= \frac{4(1+pk)^2(4k^2-1)^2}{(4k^2+1)^2}. \end{aligned}$$

Hence,

$$e = \frac{2(1+pk)(4k^2-1)}{(4k^2+1)}.$$

Then,

$$\begin{aligned} \text{the area of the right-angled triangle} &= \frac{1}{2}\frac{2(1+pk)^2(4k^2-1)^2}{(4k^2+1)^2} \cdot \frac{8k(1+pk)}{4k^2+1} \\ &= \frac{8k(4k^2-1)(1+pk)^2}{(4k^2+1)^2}. \end{aligned}$$

If we magnify the 3 sides of the triangle $\triangle BGF$ by $\frac{1}{2}\left(\frac{4k^2+1}{1+pk}\right)$ times, we have a new right-angle triangle $\triangle B'G'F'$ with 3 sides a', h', e' :

$$\begin{cases} a' = 4k^2+1 \\ h' = 4k \\ e' = 4k^2-1 \end{cases}$$

and they are positive integers and the area of $\triangle E'G'F' = 2k(4k^2-1)$ is also a positive integer. Hence, the area of this triangle is a congruent number. If we consider the general case, we still have a similar result.

Let h be the height of the triangle.

$$\begin{aligned}\frac{1}{2}\frac{2p(4k^2+m^2)}{m(4k-p)}\cdot h &= \frac{8kp(m^2+pk)}{m(4k-p)}\\ h &= \frac{8k(m^2+pk)}{(4k^2+m^2)}\\ a^2-h^2 &= e^2\\ e^2 &= \frac{4(m^2+pk)^2}{m^2}-\frac{64k^2(m^2+pk)}{(4k^2+m^2)^2}\\ &= \frac{4(m^2+pk)^2(4k^2-m^2)^2}{m^2(4k^2+m^2)^2}\\ e &= \frac{2(m^2+pk)(4k^2-m^2)}{m(4k^2+m^2)}.\end{aligned}$$

$$\text{Area of the right-angled triangle } \triangle BGF = \frac{1}{2}\frac{2(m^2+pk)(4k^2-m^2)}{m(4k^2+m^2)}\cdot\frac{8k(m^2+pk)}{4k^2+m^2} = \frac{8k(m^2+pk)^2(4k^2-m^2)}{m(4k^2+m^2)^2}.$$

If we magnify the 3 sides of the triangle $\triangle BGF$ by $\frac{1}{2}\left(\frac{4k^2+m^2}{m^2+pk}\right)$ times, we have a new right-angle triangle $\triangle B'G'F'$ with 3 sides:

$$\begin{cases} a' = \dfrac{4k^2+m^2}{m}\\ h' = 4k\\ e' = \dfrac{4k^2-m^2}{m}\end{cases}$$

are positive rational numbers and the area of $\triangle B'G'F' = \dfrac{2k(4k^2-m^2)}{m}$ is also a positive rational. Hence, if $m \mid 8k^3$, then the area of this triangle is a congruent number.

7. Existence of special forms of solutions of the Erdős-Straus Conjecture

7.1. Solutions of the Erdős-Stratus conjecture when $\mathbf{m^2 = k}$

We want to prove that to which type of prime will k, k_1, k_2 exist when $m^2 = k$.

Theorem 13. *k, k_1 ,k_2 exist when $m^2 = k$ if there exist $J \in \mathbb{N}$ such that*

$$4J-1 \mid p+1.$$

Proof. Consider $k_1 = \dfrac{k(p+1)}{4k-p}$. Since $(4k-p,k)=1$, $4k-p \mid p+1$.

We let $p = 4p'+1$ and $k = p'+J, J \in \mathbb{N}$, i.e $k_1 = \dfrac{k(p+1)}{4J-1}$.

Also, consider $k_2 = \dfrac{pk(p+1)}{4J-1} = pr_1$.

By $(4k-p,p)=1$, k_2 exist if and only if k_1 exist.

Therefore k, k_1, k_2 exist when $m^2 = k$ if and only if there exist $J \in \mathbb{N}$ s.t

$$4J-1 \mid p+1.$$

□

Theorem 14. *When $p \equiv 5 \pmod 8$, k, k_1, k_2 exist when $m^2 = k$.*

Proof. Consider $p \equiv 5 \pmod 8 \implies p+1 \equiv 6 \pmod 8$.

By direct checking $\dfrac{1}{4}\left(\dfrac{p+1}{2}+1\right)$ is an natural number.

Therefore J will exist by letting $J = \dfrac{1}{4}\left(\dfrac{p+1}{2}+1\right)$ by Theorem 13. □

7.2. Solutions of the Erdős-Stratus conjecture when $\mathbf{m^2 = 2k}$

We want to prove that to which type of prime will k, k_1, k_2 exist when $m^2 = 2k$.

Theorem 15. *k, k_1, k_2 exist when $m^2 = 2k$ if and only if $(p+2)$ contains factors in the form of $8k_1+5$ for $p \equiv 1 \pmod 8$ for $p \geq 3$.*

Proof. Consider $k_2 = \dfrac{pk}{m^2}\left(\dfrac{m^2+pk}{4k-p}\right) = \dfrac{pk(p+2)}{2(4k-p)} \implies 2 \mid k$.

Also, consider $r_1 = \dfrac{m^2+pk}{4k-p} = \dfrac{k(p+2)}{4k-p}$.

$\because (p, 4k-p) = 1$, we have

Case 1: $(4k-p) \mid 2$.

(a) $4k-p=1 \implies p = 4k-1 \implies p \equiv 3 \pmod 4$. We have a contradiction because $p \equiv 1 \pmod 4$ originally.

(b) $4k-p=2 \implies k=1$ and $p=2$.

Case 2: $4k-p \mid p+2$.

We have $p+2 = t_1(4k-p)$ for some positive integer t_1. Then,

$$p(t_1+1) = 4kt_1 - 2$$

By Theorem 14, we only consider the situation for $p \equiv 1 \pmod 8$ (explanation will be given for the rejected case $p \equiv 3 \pmod 8$ and $p \equiv 7 \pmod 8$).

We can let $p = 8p'' + 1$

$$\implies 8p" + 1 = 4k - \frac{4k+2}{t_1+1}.$$

Let $k = p" + z$.

$$\begin{aligned} 8p" + 1 &= 8p" + 8z - \frac{8(p''+z)+2}{t_1+1} \\ 8z - \frac{8(p''+z)+2}{t_1+1} &= 1 \\ 8z\left(\frac{t_1}{t_1+1}\right) &= 1 + \frac{8p''+2}{t_1+1} \\ 8z &= 1 + \frac{8p''+3}{t_1}. \end{aligned}$$

In order to satisfy $1 + \dfrac{8p''+3}{t_1} \equiv 0 \pmod 8$, we need $t_1 \equiv 5 \pmod 8$. That means $(p+2)$ must contain factors in the form of $8k_1 + 5$. □

7.3. Solutions of the Erdős-Stratus conjecture when $\mathbf{m^2 = p}$

We want to prove that to which type of prime will k, k_1, k_2 exist when $m^2 = p$.

Theorem 16. *k, k_1, k_2 exist if and only if $p+4$ contains factors in the forms of $4j+1$, $j \in \mathbb{N}$.*

Proof. Consider $k_1 = \dfrac{p(1+k)}{4k-p}$.

By $(p, 4k-p) = 1$, $4k - p \mid k+1$, i.e. $k + 1 = (4k-p)j, j \in \mathbb{N}$.

$$\implies pj + 1 = k(4j-1)$$

$$\implies k = \frac{pj+1}{4j-1} = \frac{1}{4}\left(\frac{4pj+4}{4j-1}\right) = \frac{1}{4}\left(p + \frac{p+4}{4j-1}\right).$$

By $p+4 \equiv 1 \pmod 4$ and $4j-1 \equiv 3 \pmod 4$, if j exists then $\left(p + \dfrac{p+4}{4j-1}\right) \equiv 0 \pmod 4$. Therefore $\dfrac{1}{4}\left(p + \dfrac{p+4}{4j-1}\right) \in \mathbb{N}$.

Also consider $k_2 = \dfrac{pk(1+k)}{4k-p} = kk_1$. By $(4k-p, k) = 1$, k_2 exists if and only if k_1 exists. Therefore k, k_1, k_2 exist when $m^2 = p$ if there exists $j \in \mathbb{N}$ such that $4j - 1 \mid p + 4$. □

7.4. Solutions of the Erdős-Stratus conjecture when $\mathbf{m^2 = 2p}$

We want to prove that to which type of prime will k, k_1, k_2 exist when $m^2 = 2p$.

Theorem 17. *k, k_1, k_2 exist if $p + 8$ contains factors in the form of $8j - 1$.*

Consider $k_2 = \dfrac{kp(k+2)}{2(4k-p)} \implies 2 \mid k$.

Let $k = 2k'$. Consider $k_1 = \dfrac{2p(1+k')}{4k-p}$. Since $(p, 4k-p) = 1$, we have

Case 1: $(4k-p) \mid 2$.

(a) $4k - p = 1 \implies p = 4k - 1 \implies p \equiv 3 \pmod 4$. We have a contradiction because $p \equiv 1 \pmod 4$ originally.

(b) $4k - p = 2 \implies k = 1$ and $p = 2$.

Case 2: $(4k-p) \mid (k'+1)$.

$$\begin{aligned} &\implies k' + 1 = (8k' - p)j \text{ for some positive integer } j. \\ &\implies \quad k' = \frac{pj+1}{8j-1} \\ &\implies \quad k' = \frac{1}{8}\left(p + \frac{p+8}{8j-1}\right) \end{aligned}$$

Since, we can let $p = 8p'' + 1$ for some positive integer k'',

$$k' = \frac{1}{8}\left(8p'' + 1 + \frac{8(p''+1)+1}{8j-1}\right).$$

Similarly, for $p = 8p'' + 5$,

$$k' = \frac{1}{8}\left(8p'' + 5 + \frac{8(p''+1)+5}{8j-1}\right).$$

Therefore if we assume that k' exists, k, k_1, k_2 exist if $p+8$ contains factors in the forms of $8j - 1$.

From the above, we can see some patterns of the conditions that show the existence of k, $\mathbf{k_1}$, $\mathbf{k_2}$ when $\mathbf{m^2 = hk}$ and $\mathbf{m^2 = up}$, u, p $\in \mathbb{N}$. We will do some further investigation, see appendix.

7.5. The Existence of the solutions of the Erdős Straus Conjecture

For the solutions of the Erdős Straus Conjecture, we have found that we only need to investigate $p \equiv 1 \pmod 4$. The reasons will be shown as follows:

Case 1: When $p = 4t-1$ where t is a positive integer, then we can choose $k = t \in \mathbb{N}$, and by Theorem 5,

$$k_1 = \frac{m^2+pt}{4t-(4t-1)} = m^2+pt \in \mathbb{N} \text{ where we can choose } m^2 \mid p^2t^2,$$
$$k_2 = \frac{pt}{m^2}(m^2+pt) \in \mathbb{N} \text{ for } m^2 \mid p^2t^2.$$

Hence, we have solution for $\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}$.

Note: For m^2, we can find $m^2 \mid p^2t^2$. For example, taking $m^2 = 1$, then the solution for Erdős-Straus Conjecture is as follows:

$$\begin{cases} k = k \\ k_1 = 1+pt \\ k_2 = pt(1+pt) \end{cases}.$$

Case 2: When $p = 4t-2$, but this is not a prime. This case is rejected.

Case 3: When $p = 4t-3$, we choose $k = t \in \mathbb{N}$,

$$k_2 = \frac{m^2+pt}{4t-(4t-3)} = \frac{m^2+pt}{3}.$$

Case 3(a): When $t = 3t'$, where t' is a positive integer, then $p = 4(3t')-3 = 3(4t'-1)$ is not a prime for $4t'-1 > 0$. This case is rejected.

Case 3(b): When $t = 3t'+2$, then $p \equiv 2 \pmod 3$ and $t \equiv 2 \pmod 3$.

Taking $m^2 = p$, then

$$m^2+pt = p+pt = p(1+t) \equiv p(1+2) \equiv 0 \pmod 3$$

Hence, $k_1 \in \mathbb{N}$ and also $k_2 \in \mathbb{N}$. Therefore, we have solution for

$$\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}.$$

Case 3(c): When $t = 3t'+1$, then $p \equiv 1 \pmod 3$ and $t \equiv 1 \pmod 3$.

If t has a factor b such that $b \equiv 2 \pmod 3$, then we can take $m^2 = b$.

We have $m^2 + pt \equiv 2 + (1)(1) \equiv 0 \pmod 3$.

Hence, $k \in \mathbb{N}$ and also $k_2 \in \mathbb{N}$. Therefore, we have solution for

$$\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}.$$

Theorem 18.

(a) *If $p \equiv 3 \pmod 4$, there exists a solution of Erdős-Straus Conjecture.*
(b) *If $p \equiv 1 \pmod 4$ and $p = 4t - 3$ where $t \in \mathbb{N}$, then*
   (i) *When $t = 3t' + 2$, then there exists a solution of Erdős-Straus Conjecture.*
   (ii) *When t has a factor of $(3t' + 2)$, then there exists a solution of Erdős-Straus Conjecture.*

Proof. See the above arguments. □

Theorem 19. $\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}$ *has a solution where $k = k_1 \leq k_2$ if and only if $p \equiv 3 \pmod 4$ and $k = k_1 = \frac{p+1}{2}$.*

Proof. **Prove the "only" part:**

$$\begin{aligned} & \frac{4}{p} = \frac{1}{k} + \frac{1}{k} + \frac{1}{k_2} \\ \Longrightarrow \quad & \frac{4}{p} = \frac{2}{k} + \frac{1}{k_2} \\ \Longrightarrow \quad & \frac{4}{p} = \frac{2k_2 + k}{kk_2} \\ \Longrightarrow \quad & 4kk_2 = p(2k_2 + k) \qquad (*) \\ \Longrightarrow \quad & 2 \mid k \text{ or } 2 \mid p \end{aligned}$$

If $2 \mid p \implies p = 2 \implies \frac{4}{p} = \frac{1}{1} + \frac{1}{2} + \frac{1}{2} \implies k \neq k_1$ by Appendix 11.1. Hence, we have only $2 \mid k$. Let $k = 2k'$ where k' is a positive integer.

$$\frac{4}{p} = \frac{2}{2k'} + \frac{1}{k_2},$$
$$\frac{4}{p} = \frac{1}{k'} + \frac{1}{k_2}.$$

Let $k_2 = pk_2'$ where k_2' is not divisible by p by Theorem 9. Then,

$$\begin{aligned} & \frac{4}{p} = \frac{1}{k'} + \frac{1}{pk_2'} \\ \Longrightarrow \quad & \frac{4}{p} = \frac{pk_2' + k'}{pk'k_2'} \\ \Longrightarrow \quad & 4k'k_2' = pk_2' + k' \\ \Longrightarrow \quad & k_2' \mid k'. \end{aligned}$$

Also, from (*), for $(k,p)=1$, then

$$
\begin{aligned}
& k \,|(2k_2+k) \\
\Longrightarrow\quad & k \,|2k_2 \\
\Longrightarrow\quad & 2k' \,|2k_2 \\
\Longrightarrow\quad & k' \,|k_2 \\
\Longrightarrow\quad & k' \,|pk_2' \\
\Longrightarrow\quad & k' \,|p_2' \text{ for } (k,p)=1.
\end{aligned}
$$

Hence we have $k'=k_2'$. Then,

$$
\begin{aligned}
& \frac{4}{p}=\frac{1}{k'}+\frac{1}{pk'} \\
\Longrightarrow\quad & \frac{4}{p}=\frac{p+1}{pk'} \\
\Longrightarrow\quad & 4k'=p+1 \\
\Longrightarrow\quad & p=4k'-1\equiv 3 \pmod 4.
\end{aligned}
$$

Also,

$$
\begin{aligned}
& \frac{4}{p}=\frac{1}{k'}+\frac{1}{pk'} \\
\Longrightarrow\quad & \frac{4}{p}=\frac{1}{\frac{k}{2}}+\frac{1}{\frac{pk}{2}} \\
\Longrightarrow\quad & \frac{4}{p}=\frac{2}{k}+\frac{2}{pk} \\
\Longrightarrow\quad & \frac{2}{p}=\frac{p+1}{pk} \\
\Longrightarrow\quad & 2k=p+1 \\
\Longrightarrow\quad & k=k_1=\frac{p+1}{2}.
\end{aligned}
$$

Prove the "if" part:

This is a constructive proof (This usual method can be found in some papers.)

$p\equiv 3 \pmod 4 \Longrightarrow \dfrac{(p+1)}{2}, \dfrac{p(p+1)}{4}$ are positive integers.

Also,

$$
\frac{4}{p}=\frac{1}{\frac{p+1}{2}}+\frac{1}{\frac{p+1}{2}}+\frac{1}{\frac{p(p+1)}{4}}
$$

That is the conjecture has a solution when $p\equiv 3 \pmod 4$. $\square$

Theorem 20. $\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}$ *has a solution where* $k \le k_1 = k_2$ *if and only if* $p \equiv 3 \pmod 4$ *or* $p = 2$.

Proof. **Prove the "only if" part:**

If $\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}$ has a solution where $k \le k_1 = k_2$, then

$$\begin{aligned} & k_1 = \frac{pk}{m^2} k_1 \\ \implies \quad & pk = m^2 \\ \implies \quad & k_1 = \frac{2pk}{4k-p}. \end{aligned}$$

Since $(p, 4k-p) = 1$ and $(k, 4k-p) = 1$, then $4k - p = 1$ or $4k - p = 2$.

Case 1:

$$\begin{aligned} & 4k - p = 1 \\ \implies \quad & p = 4k - 1 \\ \implies \quad & p = 3 \pmod 4. \end{aligned}$$

Case 2:

$$\begin{aligned} & 4k - p = 2 \\ \implies \quad & 2 \mid p \\ \implies \quad & p = 2. \end{aligned}$$

Prove the "if" part:

Case 1:

If $p \equiv 3 \pmod 3$, then let $p = 4k - 1$ where k is a positive integer.

Construct $k_1 = k_2 = 2pk$.

Then, $\frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2} = \frac{2p+2}{2pk} = \frac{p+1}{pk} = \frac{4k}{pk} = \frac{4}{p}$.

Hence, $\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}$ has a solution where $k \le k_1 = k_2$.

Case 2:

If $p = 2$, then by Appendix 11.1, $\frac{4}{2} = 1 + \frac{1}{2} + \frac{1}{2}$.

Hence, $\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}$ has a solution where $k \le k_1 = k_2$. □

Theorem 21. $(k_1, k_2) \neq 1$ *except* $p = 3$.

Proof.

$$\begin{aligned} k_1 &= \frac{m^2 + pk}{4k - p} \\ k_1(4k - p) - pk &= m^2 \\ k_2 &= \frac{pk}{m^2}(k_1) \\ k_2 &= \frac{pkk_1}{k_1(4k - p) - pk}. \end{aligned}$$

Assume that $(k_1, k_2) = 1$. Then $k_1 \mid k_1(4k - p) - pk$ and we have $(k_1(4k - p) - pk)$ must be divisible by all $p_i^{r_i}$ where $k_1 = p_1^{r_1} p_2^{r_2} \dots p_n^{r_n}$, $\implies p_i^{r_i} \mid pk$.

Case 1:

$p_i \mid p$, that is $p_i = p$ for some i. Therefore, $p \mid k_1$. Also, by Theorem 8, $p \mid k_2$ so $p \mid (k_1, k_2) \implies (k_1, k_2) > 1$. We have a contradiction.

Case 2:

All $p_i^{k_i} \mid k \implies k_1 \leq k$. Since $k \leq k_1$, we have $k = k_1$. By Theorem 19, we have $p \equiv 3 \pmod 4$. Let $p = 4q + 3$ where q is a non-negative integer.

Case 2(a): $q \geq 1$

By Theorem 19,

$$\begin{aligned} k_1 &= \frac{p + 1}{2} = \frac{4q + 3 + 1}{2} = \frac{4(q + 1)}{2} = 2(q + 1), \\ k_2 &= \frac{p(p + 1)}{4} = \frac{p(4)(q + 1)}{4} = p(q + 1). \end{aligned}$$

Hence, $(k_1, k_2) \geq (q + 1) > 1$. We have a contradiction.

Case 2(b): When $q = 0$, $k_1 = 2(0 + 1) = 2$, $k_2 = p(0 + 1) = p = 3$ for $q = 0$.

We have $(k_1, k_2) = 1$.

Hence, $(k_1, k_2) > 1$ except $p = 3$. □

8. Further investigation on the results obtained from some papers of the Erdős-Straus Conjecture

From some papers, we know that when the prime number $p \equiv 1^2, 11^2, 13^2, 17^2, 19^2, 23^2 \pmod{840}$ (refer to Appendix 11), we do not know whether all these prime numbers satisfy the Erdős-Straus Conjecture or not, but we make some good refinements of the above situation in this paper.

Now we consider the several cases of the existence of the solutions of Erdős-Straus Conjecture.

We assume that $p \geq 3$, otherwise $p = 2$ and we have a solution of Erdős-Straus Conjecture in Appendix 11.1.

Firstly, we consider the situation that:

$$k_1 = \frac{m^2 + pk}{4k - p}, \quad m^2 \mid pk \text{ and } (k, p) = 1.$$

However, this situation is not true for all the solutions found from the Erdős Straus Conjecture.

By Mathlab, we can find an example.

When $p = 2521$, all the solutions for $\frac{4}{2521} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}$ satisfy that pk is not divisible by m^2 but $(pk)^2$ is divisible by m^2.

k	k_1	k_2	m^2
636	69748	131876031	848
636	70588	5611746	20168
638	51997	23833534	3509
638	55462	804199	110924
644	30252	1217643	40336
652	18908	23833534	1304
658	14946	131876031	188
748	4004	42899857	176
1026	1634	55610739	76

By direct checking, all the above m^2 make the fact that pk is not divisible by m^2 but $(pk)^2$ is divisible by m^2.

Anyway, we still consider the case: $k_1 = \frac{m^2 + pk}{4k - p}$, $m^2 \mid pk$ and $(k, p) = 1$.

Firstly, by Theorem 18, we only need to consider $p \equiv -3 \pmod 4$ that is $p = 4k' - 3$ where k' is a positive integer.

Secondly, we consider the following cases.

Case 1: $m^2 \mid k$ only.

$$
\begin{aligned}
m^2 \mid k &\implies \exists r \in \mathbb{N} \text{ such that } m^2 r = k, \text{ i.e. } m^2 = \frac{k}{r} \\
&\implies k_1 = \frac{\frac{k}{r} + pk}{4k - p} = \frac{k(1+pr)}{r(4k-p)} \\
&\implies \frac{k}{r} \in \mathbb{N} \text{ and } \frac{pr+1}{4k-p} \in \mathbb{N}
\end{aligned}
$$

$$
\implies \begin{cases} k = rx & (1) \\ 1 + pr = (4k - p)y & (2) \end{cases}
$$

where x, y are positive integers.

By (2), we have

$$\frac{1+pr}{y} + p = 4k.$$

For $p \equiv -3 \pmod 4$, we have $\dfrac{1+pr}{y} \equiv 3 \pmod 4$.

Then, we let $\dfrac{1+pr}{y} = 4a + 3$ where a is a non-negative integer.

By (2), we have

$$
\begin{aligned}
4k - p &= 4a + 3 \\
4k &= 4k' - 3 + 4a + 3 \\
4k &= 4k' + 4a \\
k &= k' + a \qquad (3)
\end{aligned}
$$

From (2):

$$
\begin{aligned}
1 + pr &= (4a+3)y \\
pr &= (4a+3)y - 1 \qquad (4)
\end{aligned}
$$

Since $p \equiv -3 \pmod 4$, that is $p \equiv 1 \pmod 4$, then by (4), we have $r \equiv 3(y+1) \pmod 4$.

Let $r = 4z + 3(y+1)$ where z is a non-negative integer.

We consider $p \equiv g^2 \pmod{840}$, $g = 1, 11, 13, 17, 19, 23$, $p = 840n + g^2$ for some non-negative integers n.

Since $p = 4k' - 3$,

$$4k' - 3 = 840n + g^2$$
$$k' = 210n + \frac{g^2+3}{4}$$

By (3) and (1),

$$210n + \frac{g^2+3}{4} + a = (4z + 3(y+1))x$$
$$210n = (4z + 3(y+1))x - a - \left(\frac{g^2+3}{4}\right)$$

Since $210 = 2 \times 3 \times 5 \times 7$, then we have congruences:

$$\begin{cases} (4z+3(y+1))x - a - \left(\frac{g^2+3}{4}\right) \equiv 0 \pmod 2 \\ (4z+3(y+1))x - a - \left(\frac{g^2+3}{4}\right) \equiv 0 \pmod 3 \\ (4z+3(y+1))x - a - \left(\frac{g^2+3}{4}\right) \equiv 0 \pmod 5 \\ (4z+3(y+1))x - a - \left(\frac{g^2+3}{4}\right) \equiv 0 \pmod 7 \end{cases}$$

However, the above system of equations is difficult to be solved. Up to now, we haven't made any further investigations.

Case 2:

$$m^2 = p\left(\frac{k}{r}\right) \text{ where } r \mid k \implies k_1 = \frac{\frac{k}{r} + pk}{4k-p} = \frac{k(1+pr)}{r(4k-p)}$$
$$\implies r \mid pk \text{ and } (4k-p) \mid (1+r)$$

where $(4k-p, p) = (4k-p, k) = 1$.

Case 2A: $r = p$

Since $(4k-p) \mid (1+p) \implies (4k-p)n = 1+p$ where n is an integer.

$$(4k - 4k' + 3)n = 1 + 4k' - 3$$
$$4k = 4k' - 3 + \frac{2(2k'-1)}{n}$$
$$4k = 4k' - 3 + \frac{2k'-1}{n'}$$

where $p = 4k' - 3$ and $n = 2n'$ where n' is an integer.

We have,

$$\frac{2k'-1}{n'} = 4k'+3 \text{ for an integer } k''.$$

If we only consider the prime $p \equiv g^2 \pmod{840}$ where $g = 1,\ 11,\ 13,\ 17,\ 19,\ 23$, then

$$\begin{aligned} 840n + g^2 &= 4k' - 3 \\ k' &= 210n + \frac{g^2+3}{4} \\ 2k'-1 &= 420n + \frac{g^2+1}{2} \end{aligned}$$

where k' and n are integers.

If we want $\dfrac{2k'-1}{n'} = 4k'+3$, then $420n + \dfrac{g^2+1}{2}$ has a factor in the form $4x+3$ where x is an integer. Since $\dfrac{g^2+1}{2} \equiv 1 \pmod 4$, then we can let

$$\begin{aligned} 420n + \frac{g^2+1}{2} &= (4x+3)(4y+3) \quad \text{where } y \text{ is a non-negative integer.} \\ &= 16xy + 12x + 12y + 8 + 1 \\ &= 4(4xy + 3x + y + 2) + 1 \\ 105n &= 4xy + 3x + 3y + 2 + \frac{1-g^2}{8}. \end{aligned}$$

For $g = 1$,

$$4xy + 3x + y + 2 \equiv 0 \pmod 3 \tag{1}$$
$$4xy + 3x + y + 2 \equiv 0 \pmod 5 \tag{2}$$
$$4xy + 3x + y + 2 \equiv 0 \pmod 7 \tag{3}$$

and

$$n = \frac{\left(4xy + 3x + y + 2 + \dfrac{1-g^2}{8}\right)}{105}.$$

By using Excel, we can find all values of n when x, y are the residues under the congruent to 105. The corresponding values of m are shown below and you may refer to the excel file provided. (document name: Excel for case 2A. For the others Excel, we have also named according to their corresponding cases)

x	y	n
2	47	5
47	2	5
49	4	9
4	49	9
17	17	12
10	31	13
31	10	13
35	11	16
11	35	16
5	86	19
86	5	19
7	67	20
67	7	20
40	16	26
16	40	26
20	41	33
41	20	33
14	74	42
74	14	42
34	34	46
79	19	60
19	79	60
65	26	67
26	65	67
25	91	90
91	25	90
77	32	97
32	77	97
52	52	106
37	82	119
82	37	119
70	46	126
46	70	126
59	59	136
62	62	150
44	89	153
89	44	153
55	76	163
76	55	163
95	56	207
56	95	207

100	61	237
61	100	237
80	101	313
101	80	313
94	94	342
97	97	364
104	104	418

Case 2B: $r \mid k$ only.

Then we have $(4k-p) \mid (1+r)$ for $k_1 = \dfrac{pk(1+r)}{r(4k-p)}$.

Let

$$k = rx \tag{1}$$
$$1 + r = (4k-p)y, \tag{2}$$

where x, y are integers.

From (2),

$$\frac{1+r}{y} + p = 4k.$$

For $p \equiv -3 \pmod 4$, then $\dfrac{1+r}{y} \equiv 3 \pmod 4$. We can let $1 + r = (4a+3)y$ where a is a non-negative integer. Hence, $(4k-p) = 4a+3$.

Now we consider four cases of y under the modulus of 4 although it is not necessary to do like that.

Case 2(b)(i): $y \equiv 0 \pmod 4$.

Let $y = 4b$ where b is a positive integer. Then,

$$1 + r = (4a+3)(4b)$$
$$r = (4a+3)(4b) - 1 \tag{3}$$

Since,

$$\begin{aligned} 4k - p &= 4a + 3 \\ 4k &= p + 4a + 3 \\ 4k &= 4k' - 3 + 4a + 3 \\ k &= k' + a. \end{aligned}$$

From (1) and (3),

$$\begin{aligned} k' + a &= [(4a+3)(4b) - 1]x \\ k' &= [(4a+3)(4b) - 1]x - a. \end{aligned}$$

For $p \equiv g^2 \pmod{840}$ where $g = 1, 11, 13, 15, 17, 19, 23$, as before, we let

$$p = 840n + g$$

for some integers n.

$$\begin{aligned} 4k' - 3 &= 840n + g \\ 4k' &= 840n + g + 3 \\ k' &= 210n + \left(\frac{g+3}{4}\right) \end{aligned}$$

Then,

$$\begin{aligned} 210n + \frac{g+3}{4} &= [(4a+3)(4b) - 1]x - a \\ 210n &= [(4a+3)(4b) - 1]x - a - \frac{g+3}{4} \end{aligned}$$

Since, $210 = 2 \times 3 \times 5 \times 7$, then a, b, x must satisfy the following congruences:

$$\begin{aligned} [(4a+3)(4b) - 1]x - a - \left(\frac{g+3}{4}\right) &\equiv 0 \pmod{2} \\ [(4a+3)(4b) - 1]x - a - \left(\frac{g+3}{4}\right) &\equiv 0 \pmod{3} \\ [(4a+3)(4b) - 1]x - a - \left(\frac{g+3}{4}\right) &\equiv 0 \pmod{5} \\ [(4a+3)(4b) - 1]x - a - \left(\frac{g+3}{4}\right) &\equiv 0 \pmod{7} \end{aligned}$$

By solving these congruences, we can have

$$n = \frac{[(4a+3)(4b) - 1]x - a - \left(\frac{g+3}{4}\right)}{210},$$

where a, b, x are the residues under the module of 210. Then there are many choices of n, please refer to the excel file.

Case 2(b)(ii): $y \equiv 1 \pmod{4}$.

Let $y = 4b + 1$ where b is a positive integer. Then,

$$\begin{aligned} 1 + r &= (4a+3)(4b+1) \\ r &= (4a+3)(4b+1) - 1 \end{aligned} \tag{3}$$

Since,

$$\begin{aligned} 4k - p &= 4a + 3 \\ 4k &= p + 4a + 3 \\ 4k &= 4k' - 3 + 4a + 3 \\ k &= k' + a \end{aligned}$$

From (1) and (3),

$$\begin{aligned} k' + a &= [(4a+3)(4b+1) - 1]x \\ k' &= [(4a+3)(4b+1) - 1]x - a \end{aligned}$$

For $p \equiv g^2 \pmod{840}$ where $g = 1, 11, 13, 15, 17, 19, 23$, as before, we let $p = 840n + g$ for some integers n.

$$\begin{aligned} 4k' - 3 &= 840n + g \\ 4k' &= 840n + g + 3 \\ k' &= 210n + \frac{g+3}{4} \end{aligned}$$

Then,

$$\begin{aligned} 210n + \frac{g+3}{4} &= [(4a+3)(4b+1) - 1]x - a \\ 210n &= [(4a+3)(4b+1) - 1]x - a - \left(\frac{g+3}{4}\right) \end{aligned}$$

Since, $210 = 2 \times 3 \times 5 \times 7$, then a, b, x must satisfy the following congruences:

$$\begin{aligned} [(4a+3)(4b+1) - 1]x - a - \left(\frac{g+3}{4}\right) &\equiv 0 \pmod{2} \\ [(4a+3)(4b+1) - 1]x - a - \left(\frac{g+3}{4}\right) &\equiv 0 \pmod{3} \\ [(4a+3)(4b+1) - 1]x - a - \left(\frac{g+3}{4}\right) &\equiv 0 \pmod{5} \\ [(4a+3)(4b+1) - 1]x - a - \left(\frac{g+3}{4}\right) &\equiv 0 \pmod{7} \end{aligned}$$

By solving these congruences, we can have

$$n = \frac{[(4a+3)(4b+1) - 1]x - a - \left(\frac{g+3}{4}\right)}{210},$$

where a, b, x are the residues under the module of 210. Then there are many choices of n, please refer to the excel file.

Case 2(b)(iii): $y \equiv 2 \pmod{4}$.

Let $y = 4b + 2$ where b is a non-negative integer. Then,

$$\begin{aligned} 1 + r &= (4a + 3)(4b + 2) \\ r &= (4a + 3)(4b + 2) - 1 \end{aligned} \tag{3}$$

Since,

$$\begin{aligned} 4k - p &= 4a + 3 \\ 4k &= p + 4a + 3 \\ 4k &= 4k' - 3 + 4a + 3 \\ k &= k' + a \end{aligned}$$

From (1) and (3),

$$\begin{aligned} k' + a &= [(4a + 3)(4b + 2) - 1]x \\ k' &= [(4a + 3)(4b + 2) - 1]x - a \end{aligned}$$

For $p \equiv g^2 \pmod{840}$ where $g = 1, 11, 13, 15, 17, 19, 23$, as before, we let

$$p = 840n + g$$

for some integers n.

$$\begin{aligned} 4k' - 3 &= 840n + g \\ 4k' &= 840n + g + 3 \\ k' &= 210n + \left(\frac{g+3}{4}\right) \end{aligned}$$

Then,

$$\begin{aligned} 210n + \frac{g+3}{4} &= [(4a + 3)(4b + 2) - 1]x - a \\ 210n &= [(4a + 3)(4b + 2) - 1]x - a - \left(\frac{g+3}{4}\right) \end{aligned}$$

Since, $210 = 2 \times 3 \times 5 \times 7$, then a, b, x must satisfy the following congruences:

$$[(4a + 3)(4b + 2) - 1]x - a - \left(\frac{g+3}{4}\right) \equiv 0 \pmod{2}$$
$$[(4a + 3)(4b + 2) - 1]x - a - \left(\frac{g+3}{4}\right) \equiv 0 \pmod{3}$$
$$[(4a + 3)(4b + 2) - 1]x - a - \left(\frac{g+3}{4}\right) \equiv 0 \pmod{5}$$
$$[(4a + 3)(4b + 2) - 1]x - a - \left(\frac{g+3}{4}\right) \equiv 0 \pmod{7}$$

By solving these congruences, we can have

$$n = \frac{[(4a+3)(4b+2)-1]x - a - \left(\dfrac{g+3}{4}\right)}{210},$$

where a, b, x are the residues under the module of 210. Then there are many choices of n, please refer to the excel file.

Case 2(b)(iv): $y \equiv 3 \pmod 4$.

Let $y = 4b+3$ where b is a positive integer. Then,

$$\begin{aligned} 1 + r &= (4a+3)(4b+3) \\ r &= (4a+3)(4b+3) - 1 \end{aligned} \tag{3}$$

Since,

$$\begin{aligned} 4k - p &= 4a + 3 \\ 4k &= p + 4a + 3 \\ 4k &= 4k' - 3 + 4a + 3 \\ k &= k' + a \end{aligned}$$

From (1) and (3),

$$\begin{aligned} k' + a &= [(4a+3)(4b+3)-1]x \\ k' &= [(4a+3)(4b+3)-1]x - a \end{aligned}$$

For $p \equiv g^2 \pmod{840}$ where $g = 1, 11, 13, 15, 17, 19, 23$, as before, we let

$$p = 840n + g$$

for some integer n.

$$\begin{aligned} 4k' - 3 &= 840n + g \\ 4k' &= 840n + g + 3 \\ k' &= 210n + \left(\frac{g+3}{4}\right) \end{aligned}$$

Then,

$$\begin{aligned} 210n + \frac{g+3}{4} &= [(4a+3)(4b+3)-1]x - a \\ 210n &= [(4a+3)(4b+3)-1]x - a - \left(\frac{g+3}{4}\right) \end{aligned}$$

Since, $210 = 2 \times 3 \times 5 \times 7$, then a, b, x must satisfy the following congruences:

$$[(4a+3)(4b+3)-1]x - a - \left(\frac{g+3}{4}\right) \equiv 0 \pmod 2$$
$$[(4a+3)(4b+3)-1]x - a - \left(\frac{g+3}{4}\right) \equiv 0 \pmod 3$$
$$[(4a+3)(4b+3)-1]x - a - \left(\frac{g+3}{4}\right) \equiv 0 \pmod 5$$
$$[(4a+3)(4b+3)-1]x - a - \left(\frac{g+3}{4}\right) \equiv 0 \pmod 7$$

By solving these congruences, we can have

$$n = \frac{[(4a+3)(4b+3)-1]x - a - \left(\frac{g+3}{4}\right)}{210},$$

where a, b, x are the residues under the module of 210. Then there are many choices of n, please refer to the excel file.

Other than the Case 1 and Case 2(a) and 2(b), we have observed many solutions of Erdős-Straus Conjecture (refer to Appendix 11) are in the following forms:

Let $p = 4(3q+1) - 3$.

We only consider q is even, otherwise when q is odd then $(3q+1)$ is even such that $(3q+1)$ has 2 as a factor. By Theorem 18, p with q is odd must have a solution of Erdős Straus Conjecture.

Hence, we only consider p with q is even.

Case 3(A): We let k be even and then let $m^2 = 2p$ (it is legitimate because $m^2 \mid (pk)^2$ by Theorem 6) and let $k = (3q+1) + t$.

For k is even and q is even, then t is odd. For $k_1 = \dfrac{m^2 + pk}{4k - p}$, we have

$$k_1 = \frac{2p + pk}{4k - p} = \frac{p(2 + (3q+1) + t)}{4t + 3}$$

For k_1 is a positive integer and $4t + 3$ is not divisible by p for $(4k - p, p) = 1$ by Theorem 7 and $(4k - p) = 4t + 3$, we have

$$\frac{2 + (3q+1) + t}{4t + 3} = 3\left(\frac{q - t}{4t + 3}\right) + 1$$

is a positive integer, then $4t + 3 = 3$ or $(4t + 3) \mid (q - t)$.

However, when $4t + 3 = 3$, we have $t = 0$. Contradiction for t is odd. Hence, $(4t + 3) \mid (q - t)$.

We have, $q-t=(4t+3)x$ where x is a non-negative integer.

$$q=(4t+3)x+t=4tx+3x+t$$

Consider $p\equiv 1^2 \pmod{840}$ that is $p=840n+1$ for a positive integer.

Also $p=4(3q+1)-3$, we have $q=70n$. Hence, $n=\dfrac{4xt+3x+t}{70}$.

Now, we let $x=a$ and $t=b'$. We have, $n=\dfrac{4ab+3a+b'}{70}$ where b' id odd for t is odd.

Let $b'=2b+1$. We have

$$n=\frac{4ab+3a+b'}{70}=\frac{4a(2b+1)+3a+2b+1}{70}=\frac{8ab+7a+2b+1}{70}.$$

To solve for n, we can solve the system of congruences:

$$\begin{cases} 8ab+7a+2b+1\equiv 0 \pmod{2} \\ 8ab+7a+2b+1\equiv 0 \pmod{5} \\ 8ab+7a+2b+1\equiv 0 \pmod{7} \end{cases}$$

The solutions of n can be referred to the excel file where the solutions of n are obtained from

$$n=\frac{8ab+7a+2b+1}{70}$$

where a,b are the residues under the module of 70.

Note: Case 3(A): $m^2=2p$ is under the Case 2B (ii).

Case 3(B): Let k be even and we can let $m^2=2k$ and $k=(3q+1)+t$ where t is also odd.

For $k_1=\dfrac{m^2+pk}{4k-p}$, we have

$$\begin{aligned} k_1 &= \frac{2k+pk}{4k-p} \\ &= \frac{k(2+p)}{4k-p} \\ &= \frac{k(2+4(3q+1)-3)}{4t+3} \\ &= \frac{k(3)(4q+1)}{4t+3} \\ &= \frac{k(3)(280n+1)}{4t+3} \quad \text{where } q=70n. \end{aligned}$$

For $(k,4t+3)=1$ by Theorem 7, $\dfrac{3(280n+1)}{4t+3}$ is a positive integer.

Case 3B(i): $280n + 1$ has a factor $(4x + 1)$.

We have, $280n + 1 = (4x + 1)(4y + 1)$ where x and y are non-negative integers.

We let $3(4y + 1) = 4t + 3$ for t is also odd then y is also odd. [See reviewer's comment (2)]

Then,

$$\frac{3(280n + 1)}{4t + 3} = \frac{(4x + 1)(3)(4y + 1)}{4t + 3} = 4x + 1$$

which is a positive integer.

Hence, by $280n+1 = (4x+1)(4y+1)$ where x and y are non-negative integers and y is odd, we have

$$280n + 1 = (4x + 1)(4y + 1)$$
$$70n = 4xy + x + y$$

Then,

$$n = \frac{4xy + x + y}{70} = \frac{4a(2b + 1) + a + 2b + 1}{70} = \frac{8ab + 5a + 2b + 1}{70}$$

where $x = a$ and $y = 2b + 1$.

To solve for n, we can solve the system of congruences:

$$\begin{cases} 8ab + 5a + 2b + 1 \equiv 0 \pmod 2 \\ 8ab + 5a + 2b + 1 \equiv 0 \pmod 5 \\ 8ab + 5a + 2b + 1 \equiv 0 \pmod 7 \end{cases}$$

The solutions of n can be referred to the excel file where the solutions of n are obtained from

$$n = \frac{8ab + 5a + 2b + 1}{70}$$

where a, b are the residues under the module of 70.

Case 3B (ii): 280n + 1 has a factor $(4x + 3)$.

$280n + 1 = (4x + 3)(4y + 3)$ where x and y are non-negative integers.

We let $(4y + 3) = 4t + 3$. For t is also odd then y is also odd. [See reviewer's comment (3)]

Then

$$\frac{(280n + 1)}{4t + 3} = \frac{(4x + 3)(4y + 3)}{4t + 3} = 4x + 1$$

which is a positive integer.

Hence, by $280n+1=(4x+3)(4y+3)$ where x and y are non-negative integers and y is odd, we have

$$280n+1 = 16xy+12x+12y+9,$$
$$70n = 4xy+3x+3y+2.$$

Then,

$$n=\frac{4xy+3x+3y+2}{70}=\frac{4a(2b+1)+3a+3(2b+1)+2}{70}=\frac{8ab+7a+6b+5}{70}$$

where $x=a$ and $y=2b+1$.

To solve for n, we can solve the system of congruences:

$$\begin{cases} 8ab+7a+6b+5\equiv 0 \pmod 2 \\ 8ab+7a+6b+5\equiv 0 \pmod 5 \\ 8ab+7a+6b+5\equiv 0 \pmod 7 \end{cases}$$

The solutions of n can be referred to the excel file where the solutions of n are obtained from

$$n=\frac{8ab+7a+6b+5}{70}$$

where a, b are the residues under the module of 70.

Case 3C: Let k be even and let $m^2=\dfrac{k}{2}$ and $k=(3q+1)+t$ where t is odd.

$$\frac{\frac{k}{2}+pk}{4t+3}=\frac{\frac{k}{2}(1+2p)}{4t+3}=\frac{\frac{k}{2}(1+2(4(3q+1)-3))}{4t+3}=\frac{k}{2}\left(\frac{24q+3}{4t+3}\right)$$

For $(k,4t+3)=1$, $\dfrac{3(8q+1)}{4t+3}$ is a positive integer.

It is easy to see that $8q+1$ can be equal to $(8x+1)(8y+1)$ or $(8x+3)(8y+3)$ or $(8x+7)(8y+7)$ where x and y are non-negative integers.

But $8q+1=(8x+1)(8y+1)$ and $8q+1=(8x+3)(8y+3)$ where x and y are non-negative integers do not work now.

If $8q+1=(8x+1)(8y+1)$, then let $3(8y+1)=4t+3$. But t is odd. Contradiction.

If $8q+1=(8x+3)(8y+3)$, then let $(8y+1)=4t+3$. But t is odd. Also, we have a contradiction.

Hence, we can let $8q+1=(8x+7)(8y+7)$ where x and y are non-negative integers.

Then,

$$\begin{aligned} 560n + 1 &= 64xy + 56x + 56y + 49 \\ 70n &= 8xy + 7x + 7y + 6 \\ n &= \frac{8ab + 7a + 7b + 6}{70} \quad \text{where } a = x \text{ and } b = y. \end{aligned}$$

To solve for n, we can solve the system of congruences:

$$\begin{cases} 8ab + 7a + 7b + 6 \equiv 0 \pmod 2 \\ 8ab + 7a + 7b + 6 \equiv 0 \pmod 5 \\ 8ab + 7a + 7b + 6 \equiv 0 \pmod 7 \end{cases}$$

The solutions of n can be referred to the excel file where the solutions of n are obtained from

$$n = \frac{8ab + 7a + 7b + 6}{70}$$

where a, b are the residues under the module of 70.

For Case 3:

For the general solutions of n, we could need to amend the equations we obtain before:

Case 2B(ii):

$$n = \frac{((4a+3)b-1)c - a - \dfrac{g-3}{4}}{210} + ((4a+3)b-1)r,\ r \in \mathbb{N} \text{ for } q = 1.$$

Case 3A:

$$n = \frac{8ab + 7a + 2b + 1}{70} + (4a+1)r,\ r \in \mathbb{N}.$$

(one more general case will be

$$n = \frac{8ab + 7a + 2b + 1}{70} + (4a+1)r + (8b+7)s + 280rs,\ r,\ s \in \mathbb{N}$$

but we only consider $s = 0$, the other cases are also considered similarly.)

Case 3B(i):

$$n = \frac{8ab + 5a + 2b + 1}{70} + (4a+1)r,\ r \in \mathbb{N}.$$

Case 3B(ii):

$$n = \frac{8ab + 7a + 6b + 5}{70} + (4a+3)r,\ r \in \mathbb{N}.$$

Case 3C:

$$n = \frac{8ab + 7a + 7b + 6}{70} + (8a + 7)r, \ \ r \in \mathbb{N}.$$

Now, we will consider other forms of $p \equiv g^2 \pmod{840}$ for $g = 1, 11, 13, 17, 19, 23$.

When $p \equiv 11^2 \pmod{840}$, for $p = 4(3q + 1) - 3$, we have

$$840n + 121 = 4(3q + 1) - 3,$$
$$n = \frac{q - 10}{70}.$$

Then, the corresponding 4 cases:

Case 2B(ii):

$$n = \frac{((4a + 3)b - 1)c - a - \dfrac{g - 3}{4}}{210} + ((4a + 3)b - 1)r, \ \ r \in \mathbb{N} \text{ for } g = 11.$$

Case 3A:

$$n = \frac{8ab + 7a + 2b - 9}{70} + (4a + 1)r, \ \ r \in \mathbb{N}.$$

Case 3B(i):

$$n = \frac{8ab + 5a + 2b - 9}{70} + (4a + 1)r, \ \ r \in \mathbb{N}.$$

Case 3B(ii):

$$n = \frac{8ab + 7a + 6b - 5}{70} + (4a + 3)r, \ \ r \in \mathbb{N}.$$

Case 3C:

$$n = \frac{8ab + 7a + 7b - 4}{70} + (8a + 7)r, \ \ r \in \mathbb{N}.$$

When $p \equiv 13^2 \pmod{840}$, for $p = 4(3q + 1)) - 3$, we have

$$840n + 169 = 4(3q + 1) - 3,$$
$$n = \frac{q - 14}{70}.$$

Then, the corresponding 4 cases:

Case 2B(ii):

$$n = \frac{((4a + 3)b - 1)c - a - \dfrac{g - 3}{4}}{210} + ((4a + 3)b - 1)r, \ \ r \in \mathbb{N} \text{ for } g = 13.$$

Case 3A:

$$n = \frac{8ab + 7a + 2b - 13}{70} + (4a+1)r,\ \ r \in \mathbb{N}.$$

Case 3B(i):

$$n = \frac{8ab + 5a + 2b - 13}{70} + (4a+1)r,\ \ r \in \mathbb{N}.$$

Case 3B(ii):

$$n = \frac{8ab + 7a + 6b - 9}{70} + (4a+3)r,\ \ r \in \mathbb{N}.$$

Case 3C:

$$n = \frac{8ab + 7a + 7b - 8}{70} + (8a+7)r,\ \ r \in \mathbb{N}.$$

When $p \equiv 17^2 \pmod{840}$, for $p = 4(3q+1) - 3$, we have

$$840n + 289 = 4(3q+1) - 3,$$
$$n = \frac{q - 24}{70}.$$

Then, the corresponding 4 cases:

Case 2B(ii):

$$n = \frac{((4a+3)b-1)c - a - \dfrac{g-3}{4}}{210} + ((4a+3)b-1)r,\ \ r \in \mathbb{N} \text{ for } q = 17$$

Case 3A:

$$n = \frac{8ab + 7a + 2b - 23}{70} + (4a+1)r,\ \ r \in \mathbb{N}$$

Case 3B(i):

$$n = \frac{8ab + 5a + 2b - 23}{70} + (4a+1)r,\ \ r \in \mathbb{N}$$

Case 3B(ii):

$$n = \frac{8ab + 7a + 6b - 19}{70} + (4a+3)r,\ \ r \in \mathbb{N}$$

Case 3C:

$$n = \frac{8ab + 7a + 7b - 18}{70} + (8a+7)r,\ \ r \in \mathbb{N}$$

When $p \equiv 19^2 \pmod{840}$, for $p = 4(3q+1) - 3$, we have

$$840n + 361 = 4(3q+1) - 3$$
$$n = \frac{q-30}{70}$$

Then, the corresponding 4 cases:

Case 2B(ii):

$$n = \frac{((4a+3)b-1)c - a - \frac{g-3}{4}}{210} + ((4a+3)b-1)r, \ r \in \mathbb{N} \text{ for } q = 19.$$

Case 3A:

$$n = \frac{8ab + 7a + 2b - 29}{70} + (4a+1)r, \ r \in \mathbb{N}.$$

Case 3B(i):

$$n = \frac{8ab + 5a + 2b - 29}{70} + (4a+1)r, \ r \in \mathbb{N}.$$

Case 3B(ii):

$$n = \frac{8ab + 7a + 6b - 25}{70} + (4a+3)r, \ r \in \mathbb{N}.$$

Case 3C:

$$n = \frac{8ab + 7a + 7b - 24}{70} + (8a+7)r, \ r \in \mathbb{N}.$$

When $p \equiv 23^2 \pmod{840}$, for $p = 4(3q+1) - 3$, we have

$$840n + 529 = 4(3q+1) - 3$$
$$n = \frac{q-44}{70}$$

Then, the corresponding 4 cases:

Case 2B(ii):

$$n = \frac{((4a+3)b-1)c - a - \frac{g-3}{4}}{210} + ((4a+3)b-1)r, \ r \in \mathbb{N} \text{ for } q = 23.$$

Case 3A:

$$n = \frac{8ab + 7a + 2b - 43}{70} + (4a+1)r, \ r \in \mathbb{N}.$$

Case 3B(i):

$$n = \frac{8ab + 5a + 2b - 43}{70} + (4a+1)r,\ \ r \in \mathbb{N}.$$

Case 3B(ii):

$$n = \frac{8ab + 7a + 6b - 39}{70} + (4a+3)r,\ \ r \in \mathbb{N}.$$

Case 3C:

$$n = \frac{8ab + 7a + 7b - 38}{70} + (8a+7)r,\ \ r \in \mathbb{N}.$$

9. Investigation of the Erdős-Straus Conjecture in algebraic dimension

In this Chapter we transform the Erdős-Straus Conjecture to diophantine equations with special requirements.

Assume

$$\frac{4}{p} = \frac{1}{k} + \frac{1}{a} + \frac{1}{b}.$$

Let $4k = p + z$, i.e. $\frac{4}{p} = \frac{1}{p+z} + \frac{1}{4a} + \frac{1}{4b}$ where $a > b > k$.

$$\Longrightarrow \quad \frac{z}{p(p+z)} = \frac{1}{4a} + \frac{1}{4b} \quad \text{and} \quad z \equiv 3 \pmod 4$$

Then we have the following relationship:

$$a + b = nz \tag{1}$$

$$4ab = pn(n+z) \tag{2}$$

where n satisfy two cases: case (i) an non-negative integer or case (ii) $n = \frac{1}{m}$ where $m \mid z \in \mathbb{N}$.

Theorem 22. $(n, p) = 1$ *when n is a non-negative integer.*

Proof. We first prove (2). By Theorem 8 we know that $\frac{4ab}{p}$ is a non-negative integer (a, b are actually k_1, k_2 using the notation in chapter 5) and $(\frac{4ab}{p}, p) = 1$.

Therefore $\frac{4ab}{p} = n(p+z)$ and $(n(p+z), p) = 1$.

Then $(n, p) = 1$ when n is a non-negative integer.

Next, taking square of (1) and minus (2), we have

$$a - b = \sqrt{(zn)^2 - pm(p+z)}$$
$$a = \frac{zn + \sqrt{(zn)^2 - pm(p+z)}}{2}$$
$$b = \frac{zn - \sqrt{(zn)^2 - pm(p+z)}}{2}$$

In order to prove the existence of a, b that are integers, the necessary condition is $\sqrt{(zn)^2 - pn(p+z)}$ is a non-negative integer, i.e. $(zn)^2 - pn(p+z) = q^2$, where q is a non-negative integer.

Then we let $q = zn - t$, where t is a non-negative integer. We have

$$\begin{aligned} & -pn(p+z) = -2znt + t^2 \\ \implies & zn(2t-p) = p^2n + t^2 \\ \implies & n \mid t^2 \end{aligned}$$

where $t^2 = nt$ when n is a non-negative integer.

Continues simplifying,

$$z = \frac{p^2n + t^2}{n(2t-p)} = \frac{1}{2n}\left(t + \frac{2p^2n + pt}{2t-p}\right) = \frac{1}{2n}\left(t + \frac{1}{2}\left(p + \frac{p^2(4n+1)}{2t-p}\right)\right)$$

where $4n + 1 \in \mathbb{N}$. □

Theorem 23. *n is a non-negative integer.*

Proof. Consider case ii) $n = \dfrac{1}{m}$ where $m \mid z \in \mathbb{N}$, i.e.

$$4n + 1 = \frac{4}{m} + 1 \quad \implies \quad m = 1, 2, 4.$$

But since we have assumed $m \mid z \in \mathbb{N}$ and $z \equiv 3 \pmod 4$, m can only be 1.

Since the values of $4n + 1$ are limited, the choices of s (i.e factors of $p^2(4n+1)$) are also limited.

If we assume z exists, then we have the following cases:

Case 1: $2t - p = s$ **where** $(p, s) = 1$**.**

Then

$$t = \frac{p+s}{2}$$

$$\implies z = \frac{1}{4n}\left(2p + s + \frac{p^2(4n+1)}{s}\right) = \frac{m}{4}\left(2p + s + \frac{p^2\left(\frac{4}{m}+1\right)}{s}\right),$$

$$m = 1 \implies s = 1, 5.$$

Then when $s = 1, z = \frac{1}{4}(5p^2 + 2p + 1) = p^2 + \frac{1}{4}(p+1)^2$.

Since $p \equiv 1 \pmod 4$, $\frac{1}{4}(p+1)^2$ is odd and $p^2 + \frac{1}{4}(p+1)^2$ is even,

this contradicts with our assumption $z \equiv 3 \pmod 4$.

When $s = 5, z = \frac{1}{4}(p^2 + 2p + 5) = 1 + \frac{1}{4}(p+1)^2$.

Since $p \equiv 1 \pmod 4$, $\frac{1}{4}(p+1)^2$ is odd and $1 + \frac{1}{4}(p+1)^2$ is even,

this contradicts with our assumption $z \equiv 3 \pmod 4$.

Case 2: $2t - p = ps$ **where** $(p, s) = 1$**.**

Then

$$t = \frac{p(s+1)}{2}$$

$$\implies z = \frac{1}{4n}\left(p(s+2) + \frac{p(4n+1)}{s}\right) = \frac{m}{4}\left(p(s+2) + \frac{p\left(\frac{4}{m}+1\right)}{s}\right),$$

$$m = 1 \implies s = 1, 5.$$

Then when $s = 1, z = \frac{1}{4}(5p + 3p) = 2p$,

this contradicts with our assumption $z \equiv 3 \pmod 4$.

When $s = 5, z = \frac{1}{4}(7p + p) = 2p$,

this contradicts with our assumption $z \equiv 3 \pmod 4$.

Case 3: $2t - p = p^2 s$ **where** $(p, s) = 1$**.**

Then

$$t = \frac{p(ps+1)}{2}$$

$$\implies z = \frac{1}{4n}\left(p(ps+2) + \frac{p(4n+1)}{s}\right) = \frac{m}{4}\left(p(ps+2) + \frac{p\left(\dfrac{4}{m}+1\right)}{s}\right),$$

$$m = 1 \implies s = 1, 5.$$

Then when $s = 1, z = \frac{1}{4}(p^2 + 2p + 5) = 1 + \frac{1}{4}(p+1)^2$.

Since $\frac{1}{4}(p+1)^2$ is odd and $p^2 + \frac{1}{4}(p+1)^2$ is even,

this contradicts with our assumption $z \equiv 3 \pmod 4$.

When $s = 5, z = \frac{1}{4}(5p^2 + 2p + 1) = p^2 + \frac{1}{4}(p+1)^2$.

Since $p \equiv 1 \pmod 4$, $\frac{1}{4}(p+1)^2$ is odd and $p^2 + \frac{1}{4}(p+1)^2$ is even,

this contradicts with our assumption $z \equiv 3 \pmod 4$.

By the above discussion, there is a contradiction when we assume $n = \frac{1}{m}$ where $m \mid z \in \mathbb{N}$.

Therefore by rejecting case, n is an non-negative integer. $\square$

Now we have known that n is an non-negative integer and we have the following cases:

Case 1: $\mathbf{2t - p = s}$ **where** $\mathbf{(p, s) = 1}$

Case 2: $\mathbf{2t - p = ps}$ **where** $\mathbf{(p, s) = 1}$

Case 3: $\mathbf{2t - p = p^2 s}$ **where** $\mathbf{(p, s) = 1}$

Now we focus on solving Case 2.

Theorem 24. $s \leq 11$ *in case 2.*

Proof. $2t - p = ps$ we have

$$\begin{aligned} z &= \frac{1}{n}\left(\frac{p(s+1)}{2} + \frac{p}{2}\left(1 + \frac{4n+1}{s}\right)\right) \\ &= \frac{p}{n}\left(\frac{s+1}{2} + \frac{1}{2}\left(1 + \frac{4n+1}{s}\right)\right) \\ &= \frac{p}{n}\left(1 + \frac{1}{2}\left(s + \frac{4n+1}{s}\right)\right). \end{aligned}$$

Since $(n,p) = 1$, $n \,\bigg|\, 1 + \frac{1}{2}\left(s + \frac{4n+1}{s}\right)$.

And by $4n+1 \in \mathbb{N}$, we let $4n+1 = sr$, $s \geq r$, where $r \in \mathbb{N}$.

$$\implies n = \frac{sr-1}{4} \implies \frac{sr-1}{4} \,\bigg|\, 1 + \frac{s+r}{2}$$

Then,

$$\begin{aligned} 1 + \frac{s+r}{2} &\geq \frac{sr-1}{4} \\ \implies \quad 4 + 2(s+r) &\geq sr - 1 \\ \implies (s-2)(r-2) &\geq 9 \end{aligned}$$

Therefore by $s \geq r$, $s \leq 11$. □

Since the values of s is limited for all primes, we can find the existence of solutions of the Erdős-Stratus Conjecture in this case by direct checking (i.e. The method used in Theorem 23). Therefore, we should focus on the other two cases.

The remaining two cases are complicate to solve, and until now we still don't have remarkable result. But we have further discussion, see Appendix.

10. Conclusion

Here we list out the results of the report.

On Chapter 4:

1. Given that $n \in \mathbb{N}$, all integral solutions (x,y) of $\frac{1}{n} = \frac{1}{x} + \frac{1}{y}$ are given by

$$x = n + s \quad \text{and} \quad y = n + \frac{n^2}{s},$$

i.e.

$$\frac{1}{n}=\frac{1}{n+s}+\frac{1}{n+\dfrac{n^2}{s}} \quad \text{where } s\in\mathbb{N} \text{ and } \frac{n^2}{s}\in\mathbb{N}.$$

2. $\dfrac{3}{n}=\dfrac{1}{x}+\dfrac{1}{y}$ exists for some $x,y\in\mathbb{N}$ if and only if

$$\text{(i) } n=3k \text{ or (ii) } n=3k+2 \text{ or (iii) } n=3k+1$$

where there exists a positive integer $f\mid n$ such that $f\equiv 2 \pmod 3$.

On the Erdős-stratus Conjecture:

1. $\dfrac{4}{p}=\dfrac{1}{k}+\dfrac{1}{k_1}+\dfrac{1}{k_2}$ where k,k_1,k_2 are positive integers if and only if

$$\begin{cases} k=k \\ k_1=\left(\dfrac{m^2+pk}{4k-p}\right) \\ k_2=\dfrac{pk}{m^2}\left(\dfrac{m^2+pk}{4k-p}\right) \end{cases}$$

where k,k_1,k_2 are positive integers and $m^2>0$.

Using the notation above, the properties of k,k_1,k_2,m^2 are as follows:

2. If

$$\begin{cases} k=k \\ k_1=\left(\dfrac{m^2+pk}{4k-p}\right) \\ k_2=\dfrac{pk}{m^2}\left(\dfrac{m^2+pk}{4k-p}\right) \end{cases}$$

where k,k_1,k_2 are positive integers, then $m^2\in\mathbb{N}$ and $m^2\mid p^2k^2$.

3. $k<p$ and k is not divisible by p.

4. k_2 is divisible by p.

5. None of k,k_1,k_2 can be divisible by p^2.

6. The bounds of k, k_1, k_2 are
$$\begin{cases} \frac{1}{4}p \leq k \leq \frac{3}{4}p \\ k_1 \leq \frac{3}{2}p^2 \\ k_2 \leq \frac{9}{4}p^4 \end{cases}.$$

About the existence of solutions of The Erdős-stratus Conjecture:

7. When $m^2 = k$, solutions of The Erdős-stratus Conjecture exist if and only if $(p+1)$ contains factors in the form of $4J-1$.

8. When $m^2 = 2k$, solutions of The Erdős-stratus Conjecture for $p \equiv 1 \pmod 8$ exist if and only if $(p+2)$ contain factors in the form of $8k_1+5$.

9. When $m^2 = p$, solutions of The Erdős-stratus Conjecture exist if and only if $p+4$ contains factors in the forms of $4j-1$.

10. When $m^2 = 2p$, solutions of The Erdős-stratus Conjecture exist if and only if $p+8$ contains factors in the forms of $8j-1$.

11. When $p = 4t-1$ where t is a positive integer, we have a solution for
$$\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}.$$

12. When $p = 4t-3$, where t is a positive integer, we have a solution for
$$\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2} \text{ when}$$
Case (i) $t = 3t' + 2$;
Case (ii) $t = 3t' + 1$, t has a factor b such that $b \equiv 2 \pmod 3$, excluding the case $t = 3t'$.

13. $\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}$ has a solution where $k = k_1 \leq k_2$ if and only if $p \equiv 3 \pmod 4$.

14. $\frac{4}{p} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}$ has a solution where $k \leq k_1 = k_2$ if and only if
$$p \equiv 3 \pmod 4 \text{ or } p = 2.$$

15. $(k_1, k_2) \neq 1$ except $p = 3$.

16. From solutions for some special forms we can discuss that for many values of n checked, e.g. for $1 \leq n \leq 3000$, if $p = 840n + 1$ is prime, n must satisfy one of the Case 2b(ii), 3A, 3B, 3C. We hope that all these 4 cases can cover all

the values of n such that $p = 840n + 1$ is prime.

Also, now we are investigating $p = 840n + g^2$ such that it is a prime. Here $g = 11,\ 13,\ 17,\ 19,\ 23$ are not checked.

On further applications,

17. If the solution of k, k_1, k_2 make $m \in \mathbb{N}$ and $m \mid 2pk$, then we can form a Herion triangle for

$$\begin{cases} s = 8m\left(\dfrac{k_2}{p}\right) \in \mathbb{N} \\ \Delta = 8mk_2 \in \mathbb{N} \\ c = 2\left(m + \dfrac{pk}{m}\right) \in \mathbb{N} \\ b = \dfrac{2k(4m^2 + p^2)}{m(4k - p)} \in \mathbb{N} \\ a = \dfrac{2p(4k^2 + m^2)}{m(4k - p)} \in \mathbb{N} \end{cases}$$

18. The Herion triangle formed in (17) cannot be a rational triangle.

11. Appendix

11.1. There is only one solution of $\frac{4}{2} = \frac{1}{k} + \frac{1}{k_1} + \frac{1}{k_2}$ where $k \le k_1 \le k_2$ and $k = 1$, $k_1 = k_2 = 2$.

Proof. Firstly, $k < 2$, otherwise if $k \ge 2$ then $\frac{4}{2} \le \frac{1}{2} + \frac{1}{2} + \frac{1}{2}$, and we have $2 \le \frac{3}{2}$. Contracdtion.

Hence, we have $k = 1$.

Secondly, $\frac{4}{2} = \frac{1}{1} + \frac{1}{k_1} + \frac{1}{k_2} \implies 1 = \frac{1}{k_1} + \frac{1}{k_2}$. We have $k_1 < 3$.

Otherwise if $k_1 \ge 3$ then $1 \le \frac{1}{3} + \frac{1}{3}$, and we have $1 \le \frac{2}{3}$. Contradiction.

Thirdly, if $k_1 = 1$, then $1 = \frac{1}{1} + \frac{1}{k_2} \implies 0 = \frac{1}{k_2}$. Contradiction. Hence, $k_1 = 2 \implies k_2 = 2$. □

11.2. when the prime number $p \equiv 1^2, 11^2, 13^2, 17^2, 19^2, 23^2 \pmod{840}$, we do not know whether all these prime numbers satisfy the Erdős-Straus Conjecture(Ch.8)

Here we provide the proof from [1] as a reference.

Here we first set up a equation

$$na + b + c = 4abcd \tag{1}$$

Dividing both sides by $abcdn$, we obtain

$$\frac{4}{n} = \frac{1}{bcd} + \frac{1}{nabd} + \frac{1}{nacd}$$

Then by

- letting $a = 2, b = 1, c = 1$, from (1) we get $n = 4d - 1$,
- letting $a = 1, b = 1, c = 1$, from (1) we get $n = 4d - 2$,
- letting $a = 1, b = 1, c = 2$, from (1) we get $n = 8d - 3$,
- letting $a = 1, b = 1, d = 1$, from (1) we get $n = 3c - 1$.

When $n = 4$, $1 = \frac{1}{3} + \frac{1}{3} + \frac{1}{3}$.

When $n = 3$, $\frac{4}{3} = \frac{1}{3} + \frac{1}{2} + \frac{1}{2}$.

From the above we know that the Erdős-Straus Conjecture is true except

$$n \equiv 1 \pmod{24}.$$

Similarly,

- letting $a = 1, b = 1, d = 2$, from (1) we get $n = 7c - 1$,
- letting $a = 1, b = 2, d = 2$, from (1) we get $n = 7c - 2$,
- letting $a = 2, b = 1, d = 1$, from (1) we get $2n = 7c - 1$.
  Let $c = 2t - 1$, then $n = 7t - 4$.

Also, $\frac{4}{7} = \frac{1}{2} + \frac{1}{28} + \frac{1}{28}$ and we know the solutions of the Erdős-Straus Conjecture exist except for $n \equiv 1 \pmod{7}$, $n \equiv 2 \pmod{7}$, $n \equiv 4 \pmod{7}$.

Similarly,

- letting $a = 1, b = 2, d = 2$, from (1) we get $n = 15c - 2$,
- letting $a = 2, b = 1, d = 2$, from (1) we get $2n = 15c - 1$.
  Let $c = 2t - 1$, then $n = 15t - 8$.

Also, $\frac{4}{5} = \frac{1}{2} + \frac{1}{5} + \frac{1}{10}$ and we know the Erdős-Straus Conjecture is true except for $n \equiv 1, 2, 3, 4, 6, 8, 9, 11, 12, 14 \pmod{15}$.

From the above we know that the Erdős-Straus Conjecture is true except

$$n \equiv 2, 0 \pmod{3}$$

and not true except

$$n \equiv 1, 4 \pmod{5}.$$

Sumarise the above, we have proved the Erdős-Straus Conjecture is true except

$$n \equiv 1 \pmod{24} \quad \text{or} \quad n \equiv 1 \pmod{7}, \quad n \equiv 2 \pmod{7}, \quad n \equiv 4 \pmod{7}$$
$$\text{or} \quad n \equiv 1, 4 \pmod{5}.$$

Therefore we have the following 6 cases:

$$n \equiv 1 \pmod{24} \text{ and } n \equiv 1 \pmod{7} \text{ and } n \equiv 1 \pmod{5};$$
$$n \equiv 1 \pmod{24} \text{ and } n \equiv 1 \pmod{7} \text{ and } n \equiv 4 \pmod{5};$$
$$n \equiv 1 \pmod{24} \text{ and } n \equiv 2 \pmod{7} \text{ and } n \equiv 1 \pmod{5};$$
$$n \equiv 1 \pmod{24} \text{ and } n \equiv 2 \pmod{7} \text{ and } n \equiv 4 \pmod{5};$$
$$n \equiv 1 \pmod{24} \text{ and } n \equiv 4 \pmod{7} \text{ and } n \equiv 1 \pmod{5};$$
$$n \equiv 1 \pmod{24} \text{ and } n \equiv 4 \pmod{7} \text{ and } n \equiv 4 \pmod{5}.$$

Since $(24, 7) = (5, 7) = (24, 5) = 1$, by Chinese remainder theorem, we have

$$n \equiv 1, 121, 169, 361, 529 \pmod{840}.$$

11.3. Mathlab program for finding the solutions of Erodos-Straus Conjecture

Referred to the paper [3].

11.4. Solutions in the form $\mathbf{m^2 = hk}$, $\mathbf{m^2 = up}$ where $(\mathbf{h}, \mathbf{p}) = (\mathbf{u}, \mathbf{p}) = \mathbf{1}$ if the solutions of Erdő-Straus Conjecture exist (Ch.7)

Here we consider the general case $m^2 = hk, m^2 = up$, where $(h, p) = (u, p) = 1$.

11.4.1. The case $\mathbf{m^2 = hk}$

Consider $k_2 = \dfrac{pk(p+h)}{h(4k-p)}$.

Since $(h, h+p) = 1$, $h \mid k$ if we want k_2 exist, i.e. $k = hk'$ and $4k - p \mid k'(h+p)$.

Also consider $k_1 = \dfrac{k(p+h)}{4k-p} = h\left(\dfrac{k'(p+h)}{4k-p}\right)$, k_1 exist if k_2 exist.

Therefore our target is to solve $4k-p \mid k'(h+p)$.

We let

$$t = \frac{k'(p+h)}{4k-p} = \frac{1}{4}\left(\frac{4hk'+4pk'}{4hk'-p}\right) = \frac{1}{4}\left(1+\frac{4pk'+p}{4hk'-p}\right) = \frac{1}{4}\left(1+\frac{p(4k'+1)}{4hk'-p}\right).$$

By $(p, 4hk'-p) = 1$, we have $4hk'-p \mid 4k'+1$ and $\dfrac{p(4k'+1)}{4hk'-p} \equiv 3 \pmod 4$.

Continue Simplifying,

$$t = \frac{1}{4}\left(1+\frac{p(4k'h+h)}{h(4hk'-p)}\right) = \frac{1}{4}\left(1+\frac{p}{h}\left(1+\frac{p+h}{4hk'-p}\right)\right).$$

Since $\dfrac{p(4k'+1)}{4hk'-p}$ is an natural number, $1+\dfrac{p+h}{4hk'-p}$ is also an natural number.

In other words, if there exist h and k' such that $4hk'-p \mid p+h$, the solution of the Erdős-Straus Conjecture for this prime exists in the form $m^2 = hk$. For Example, when we put $h = 1$, we exactly get the same result as Theorem 14. But until now we are still trying to observe the patterns in this form of solutions

11.4.2. The case $m^2 = up$

Consider $k_1 = \dfrac{p(k+u)}{4k-p}$.

By $(p, 4k-p) = 1$ we have $4k-p \mid k+u$.

Also consider $k_2 = \dfrac{kp(k+u)}{u(4k-p)}$, then we have $u \mid k^2$.

Since we match up these two relationships, we assume $u \mid k$, i.e. $k = vu$.

Hence we have $k_1 = \dfrac{pu(v+1)}{4uv-p}$.

By $(4uk-p, u) = 1$, we have $4uv-p \mid v+1$, i.e. $v+1 = (4uv-p)J,\ J \in \mathbb{N}$

$$pJ+1 = v(4vJ-1)$$

$$\implies \qquad v = \frac{pJ+1}{4uJ-1} = \frac{1}{4u}\left(\frac{4upJ+u}{4uJ-1}\right) = \frac{1}{4u}\left(p+\frac{p+4u}{4uJ-1}\right).$$

Since $\dfrac{pJ+1}{4uJ-1}$ is an natural number, $\dfrac{p+4u}{4uJ-1}$ is also an natural number.

In other words, if there exist u and J such that $4uJ-1 \mid p+4u$, the solution of the Erdős-Straus Conjecture for this prime exist in the form $m^2 = up$. For Example, when we put $h = 1$, we exactly get the same result as Theorem 16. But until now we are still trying to observe the patterns in this form of solutions.

11.5. Further discussion on Case 1: $\mathbf{2t - p = s}$ where $(\mathbf{p}, \mathbf{s}) = \mathbf{1}$ (Ch.9)

From the discussion in Ch.9, we know that $t^2 = nt'$ when n is a non-negative integer. To match up the other conditions, we try to let $t = nv_1$.

Then $p + s = 2t = 2nv$. Also, $s \mid 4n + 1$.

In other words, $2nv_1 - p \mid 4n + 1$.

Therefore we let $y = \dfrac{4n+1}{2nv_1 - p}$, where x is an natural number.

Then,

$$\begin{aligned} py + 1 &= 2nyv_1 - 4n \\ \Longrightarrow \quad py + 1 &= 2n(yv_1 - 2) \\ \Longrightarrow \quad 2n &= \frac{py+1}{yv_1 - 2}. \end{aligned}$$

Here we let $p = Lv_1 + I, I, L \in \mathbb{N}$, i.e.

$$\begin{aligned} 2n &= \frac{(Lv_1 + I)y + 1}{yv_1 - 2} \\ &= L + \frac{Iy + 2L + 1}{yv_1 - 2} \\ &= L + \frac{1}{v_1}\left(\frac{Iv_1 y + 2v_1 L + v_1}{v_1 y - 2}\right) \\ &= L + \frac{1}{v_1}\left(I + \frac{2p + v_1}{v_1 y - 2}\right) \end{aligned}$$

and

$$\frac{2p + v_1}{v_1 y - 2} \in \mathbb{N}.$$

Therefore, if there exist v_1, y such that $v_1 y - 2 \mid 2p + v_1$, then the solution of the Erdős-Straus Conjecture exist for this prime. Until now we are still observing the patterns.

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Reviewer's Comments

This paper may be too long and contain too many theorems. It is better to rewrite some theorems as lemmas or claims. There are lots of errors in this paper and the following is an incomplete list of corrections and stylistic suggestions.

1. The reviewer has comments on the wordings, which have been amended in this paper.
2. Rewrite this line as "Let $3(4y + 1) = 4t + 3$. Then t and y have the same parity."
3. Rewrite this line as "Let $4y + 3 = 4t + 3$. Then $t = y$."