Hang Lung Mathematics Awards
Vol. 8 (2018), pp. 1–25
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ON THE TRAPEZOIDAL PEG PROBLEM AMONG CONVEX CURVES

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Abstract. The Trapezoidal Peg Problem, as one of the generalizations of the famous Square Peg Problem, asks when a prescribed trapezoid can be inscribed in a given Jordan curve. We investigated a possible approach towards the problem by first weakening the similarity condition, in which we have shown that for any trapezoid, some classes of convex curves can actually inscribe, up to two kinds of weaker forms of similarity, infinitely many trapezoids. Our main theorem further analyzed the properties of one of these infinite family of trapezoids, and showed that any given trapezoid can be uniquely inscribed in any strictly convex C^1 curve, which we named 'oval', up to only translation and a kind of transformation, which we called 'stretching', but without rotation, and the resulting trapezoid moves continuously when the given trapezoid rotates. Through this, we consequently obtained a necessary and sufficient condition for an oval to inscribe an arbitrary trapezoid up to similarity, which could be taken as an answer to the problem among ovals. Some other variations are also discussed.

1. Introduction

The Trapezoidal Peg Problem was proposed as a generalization of the famous Square Peg Problem (also known as the Inscribed Square Problem) formulated by Otto Toeplitz in the year 1911. Similar to the classic Square Peg Problem, the Trapezoidal Peg Problem asked for the existence of an inscribed trapezoid which is similar to a prescribed trapezoid in any Jordan curve, that is, continuous simple closed curve on $\mathbb{R}^2$. The problem appears to be even harder compared to the classic case, since even the case for piecewise linear curves still remains unsolved according to [9]. However, it occurs that some of the methodology used in the previous works towards the classic problem might also be useful in some special cases of our generalized version.

1.1. The classic Square Peg Problem

In 1911, Otto Toeplitz formulated the Square Peg Problem in [13], the conjecture was stated as follows:

Conjecture 1. *(The Square Peg Problem). Every continuous simple closed curve in the plane $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ contains four points that are the vertices of a square.*

A continuous simple closed curve on the plane is also called a Jordan curve. Topologically, such a curve could also be viewed as a embedding from $\mathbb{S}^1$ to $\mathbb{R}^2$. Note that the conjecture do not require the entire square to be inscribed inside the curve, or else there will be some obvious counterexamples like Figure 1.

So far, even if there are already more than one hundred years after the time when Conjecture 1 was stated, it still remains unsolved. [See reviewer's comment (2)] However, although the full conjecture seems to be extraordinarily hard, some special cases have already been tackled by mathematicians.

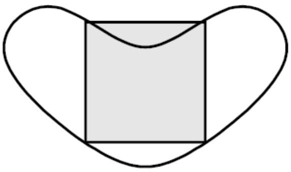

FIGURE 1. A counterexample

Most of the known solutions to the special cases of Conjecture 1 based on topological ideas, nevertheless, some geometrical reasonings could also be found. Curiously, the majority of them were formulated in the early years, or even before the conjecture was properly proposed. Our work, however, is mainly based on a geometric point of view.

It seems that Otto Toeplitz himself never published a proof for any of the special cases. The forerunner of the problem, Arnold Emch, who was suggested the problem by Kempner according to [2], proved the case for smooth enough convex curves in [3], where the main idea of Chapter 3 of this paper came from. Emch further amended his method and achieved a weaker smoothness condition on convex curves two years later in [2], and one more year later, he finally showed the case for curves that are piecewise analytic with only finitely many inflection points and other singularities where the left and right sided tangents at the finitely many non-smooth points exist in [4]. The case for general convex curves is actually a simple corollary of Fenn's table theorem [5], which is proved later in 1970. There are also many other contributions to different special cases of the problem through years accroding to a recent survey [8], which selected ones are listed below: [See reviewer's comment (3)]

- Schnirelman [11] proved the case for a class of curves that is slightly larger than C^2.
- Stromquist [12] tackled the problem for locally monotone Jordan curves.
- Makeev [6] coped with star-shaped C^2 curves that intersect every circle in at most 4 points (more generally he proved not only the square case, but the case for every cyclic quadrilaterals for such curves, see below)
- Matschke, the author of the survey, solved the problem in his PhD thesis [7] for a technical open and dense class of curves and for continuous curves in certain bounded domain.

Terence Tao [10] also looked into the problem recently. In his paper, he augments the homological approach by introducing certain integrals associated to the curve. This approach is able to give positive answers to the square peg problem in some new cases, for instance if the curve is the union of two Lipschitz graphs that agree at the endpoints, and whose Lipschitz constants are strictly less than one. This is believed to be the latest result on the problem.

1.2. The generalized problem

Many generalized versions of Conjecture 1 exists, where the generalization mainly focused on other shapes which are conjectured to be able to inscribe in some Jordan curves. [See reviewer's comment (4)] Since the circle is obviously a Jordan curve, the shape must be cyclic.

Conjecture 2. *(The Cyclic Quad Peg Problem). Let Q be a cyclic quadrilateral. Then any C^1 embedding $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ admits an orientation preserving similarity transformation that maps the vertices of Q into γ.*

In this case, the general assertion to continuous curves is actually false since, observed by Pak [9], other kinds of cyclic quadrilaterals except isosceles trapezoids cannot be inscribed in very thin triangles. However, the case for smooth curves is still conjectured. [See reviewer's comment (5)]

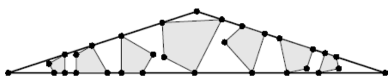

FIGURE 2. Counterexample of general cyclic quadrilaterals

Makeev [6], as said above, proved for star-shaped C^2 curves that intersect every circle in at most 4 points. [See reviewer's comment (6)] A recently submitted paper [1] to ArXiv claimed to tackle the case for any smooth convex Jordan curves, but it have not finished being reviewed. [See reviewer's comment (7)]

The full generality of continuity hence go to the family of isosceles trapezoids. In particular, Pak proposed the following conjecture:

Conjecture 3. *(The Isosceles Trapezoidal Peg Problem). Let T be an isosceles trapezoid, then any Jordan curve γ inscribes a trapezoid that is similar to T.*

In particular, Pak proposed the conjecture for piecewise linear curves. This conjecture, according to Pak, has not been proved or disproved so far.

It is clear that we cannot directly further generalize Conjecture 3 to arbitrary trapezoids, since only isosceles trapezoids can be inscribed in a circle. However, it remains to be justified on the types of curves can some prescribed trapezoid be inscribed, in which it can be formulated as the following problem, which is what this paper focuses on. [See reviewer's comment (8)]

Problem 4. *(The Trapezoidal Peg Problem). Let T be a prescribed trapezoid. Find a (meaningful) condition for a Jordan curve $\gamma : \mathbb{S}^1 \to \mathbb{R}^2$ to inscribe T.*

1.3. Our work

One of the corollaries, Corollary 29, of our main result, Theorem 14 dealt with the case when we restrict to C^1 strictly convex curves in Problem 4, where we have obtained a necessary and sufficient condition for the trapezoid to be inscribed among this class of curves.

To formulate our result, we present an idea of decomposing the condition of similarity in tackling the inscribing problem, which is inspired by the attempt of Emch [3] in solving the Square Peg Problem. Specifically, we dealt with the variation of the problem in another direction, in which we tried to see what we can say about the statement if the similarity condition is weakened. In particular, we dissolve the similarity condition between trapezoids into three components, and try to see the result when some of them are not fulfilled. Surely, once we have weaken this condition, then we are not satisfiedd with only the usual enquiry for the inscribing condition. In fact, in our classes of curves, our result can deduce that there can be infinite number of trapezoids inscribed in the given curve, up to the weakened notions of similarity. Even more interesting than that is that the properties of this infinite family of weakly inscribed trapezoids finally gives the clue on whether a prescribed arbitrary trapezoid can be inscribed.

We shall first discuss some of the useful and essential facts in our methodology, and give an example of to which degree we could generalize the problem in terms of the number of inscribed trapezoids in Chapter 2. And in Chapter 3, we shall give our main theorem, which gives a quite intriguing result and many interesting corollaries without weakening a great deal on similarity, including our formulation of equivalent condition for a prescribed trapezoid to be inscribed.

2. A taste of the idea

Before presenting our main theorems, let us just take a first look at the idea of weakening the notion of similarity. As one can see, this is quite hard to be done on general shapes on the Euclidean plane, but due to the fact that trapezoids have quite a number of beautiful and useful properties given by its opposite parallel sides, the idea could be put into practice.

2.1. Introduction

Clearly, our desired weak forms of similarity should be implied by similarity. And our idea in between is to looking for the "component" which similarity between trapezoids actually consists of. [See reviewer's comment (9)] Namely, since we have

Lemma 5. *Let $ABCD$ be a trapezoid with $AB//CD$. Let O be the intersection of AC and BD, then $AO : AC = BO : BD$.*

Proof. Since $AB//CD, \angle BAO = \angle DCO, \angle ABO = \angle CDO$, which gives us the similarity $\triangle ABO \sim \triangle CDO$, hence $AO : OC = BO : OD$, and we shall have $AO : AC = BO : BD$. □

We could make use of the following theorem:

Theorem 6. *Let $ABCD$ and A'B'C'D' be two trapezoids with $AB||CD, A'B'||C'D'$ and $AC \cap BD = O, A'C' \cap B'D' = O'$. Then $ABCD \sim A'B'C'D'$ if and only if the following are satisfied:*

(1) $AO : AC = A'O' : A'C'$
(2) $\angle AOB = \angle A'O'B'$
(3) $AC : BD = A'C' : B'D'$

Proof. If $ABCD \sim A'B'C'D'$, then the three conditions are obviously satisfied. So the remaining is to prove the sufficiency.

If $AC = A'C'$, by (3) we know $BD = B'D'$, and by (1) we further get $A'O' = AO$, and $B'O' = BO$ by Lemma 5. Therefore $\triangle AOB \cong \triangle A'O'B'$, $\triangle COD \cong \triangle C'O'D'$ by (2), thus $ABCD = A'B'C'D'$. Otherwise we could always scale $ABCD$ by the ratio $A'C'/AC$ to get a new trapezoid $A''B''C''D''$. One could check that $A'B'C'D'$ and $A''B''C''D''$ satisfies condition (1), (2) and (3), and by our assumption, $ABCD \sim A''B''C''D'' \cong A'B'C'D'$. □

Remark 7. *None of these three conditions is redundant. As in Figure 3, with respect to T, the non-similar trapezoids T_1, T_2, T_3 only missed out condition (1), (2), (3) respectively.*

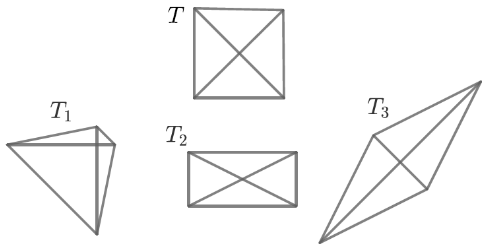

FIGURE 3. Counterexamples

While weakening similarity between trapezoids in our work, we keep condition (1), but remove condition (2) and (3) alternatively to achieve what we want. In the following sections in this chapter, we will investigate some result when we only need condition (1) to be true. To begin with, we shall come up with a few definitions and lemmas.

2.2. Definitions and lemmas

We shall first give some characterization of such a weakened similarity.

Definition 8. *Let $ABCD$ be a trapezoid, such that $AB||CD$ and $AB \leq CD$. Let $O = AC \cap BD$ be the intersection points of the diagonals of the trapezoid. Then we define the cut-ratio t of $ABCD$ by the equation $t = AO : AC = BO : BD$, which is well-defined by Lemma 5. Two trapezoids with the same cut-ratio (i.e. condition (1) is true) are said to be quasi-similar.*

It is worth to note that, for a non-degenerate trapezoid, $0 < t \leq 1/2$.

Moreover, to come up with our theorem, we shall recall the notion and an elementary property of homothety.

Definition 9. *A homothety is a transformation determined by a point $\overrightarrow{s}$ called its center and a nonzero number t called its ratio, which sends $\overrightarrow{m} \mapsto \overrightarrow{s} + t(\overrightarrow{m} - \overrightarrow{s})$.*

In particular, for any set of points forming a shape, the image of it upon some homothetic transformation with $t \neq 0$ is similar to it. Our proof basically based on this fact.

We have two more lemmata:

Lemma 10. *Let $X \subset \mathbb{R}^n$ be a convex set, then the image of any homothetic transformation centered at any point $\overrightarrow{x} \in X$ with ratio $0 < t < 1$ is contained in X.*

Proof. By definition, the image of any point $\overrightarrow{m} \in X$, i.e. $(1-t)\overrightarrow{x} + t\overrightarrow{m}$, is an element of X. □

We denote the area enclosed by a Jordan curve J on the plane by $e(J)$. [See reviewer's comment (10)]

Lemma 11. *Let γ_1, γ_2 be two Jordan curves on the plane such that neither $e(\gamma_1) \subset e(\gamma_2)$ nor $e(\gamma_2) \subset e(\gamma_1)$. If $\gamma_1 \cap \gamma_2 = \emptyset$, then $e(\gamma_1) \cap e(\gamma_2) = \emptyset$.*

Proof. We shall argue by means of contradiction. Suppose $\gamma_1 \cap \gamma_2 = \emptyset$, but $e(\gamma_1) \cap e(\gamma_2) \neq \emptyset$, then there exists $A \in \gamma_1$ such that $A \in e(\gamma_2)$. By the fact that neither $e(\gamma_1) \subset e(\gamma_2)$ nor $e(\gamma_2) \subset e(\gamma_1)$, we know that $\exists B \in \gamma_1, B \notin e(\gamma_2)$. By continuity γ_1 cuts through γ_2, which is a contradiction. □

Our main theorem concerning quasi-similarity would be on convex Jordan curves. It seems obvious that any curve of this kind can inscribe a prescribed trapezoid, up to quasi-similarity. However, we can actually show that the number of such trapezoids is essentially infinite.

2.3. The theorem

We shall prove the following theorem.

Theorem 12. *Let T be a trapezoid. Then any convex Jordan curve $\gamma : \mathbb{S}^1 \hookrightarrow \mathbb{R}^2$ inscribes infinite number of trapezoids which is quasi-similar to T.*

Proof. We denote the homothetic transformation centered at $\overrightarrow{s}$ with ratio t by $h[\overrightarrow{s}, t]$. And for nonzero t, we call $h[\overrightarrow{s}, 1/t]$ the *inverse* homothetic transformation of $h[\overrightarrow{s}, t]$.We know that, for any point $\overrightarrow{p} \in \mathbb{R}^2$

$$h[\overrightarrow{s}, t] \circ h[\overrightarrow{s}, 1/t](\overrightarrow{p}) = (1-t)\overrightarrow{s} + t\left(\left(1 - \frac{1}{t}\right)\overrightarrow{s} + \frac{1}{t}\overrightarrow{p}\right) = \overrightarrow{p}$$

and vice versa. Hence, we know that the homothetic transformation from any point set to its image is a bijection.

Now, we shall show that for any fixed $0 < t \leq 1/2$, we could find four points in γ such that they form a trapezoid with cut-ratio t, and the theorem is proved.

Choose any point $X \in \gamma$, we define the auxiliary curve γ_X of X to be the image of the homothetic transformation $h[X, t] : \gamma \to \mathbb{R}^2$. [See reviewer's comment (11)] It is easy to see that γ_X is a convex Jordan curve as well for any $X \in \gamma$. Indeed, it is, by definition, similar to γ. We let P be the collection

$$\{\gamma_X \subset \mathbb{R}^2 : X \in \gamma\}$$

Firstly, we shall show that there exist infinitely many distinct pairs of elements in P such that they have a non-empty intersection. Let the subset $K \subset P$ be the set of curves in P such that they have no intersection with any other curves in P. [See reviewer's comment (12)]

Now, let $\mu(D)$ be the set of points of $D \subset \mathbb{R}^2$. We know that $\mu(e(\gamma))$ is finite. And we know by property of homothety, or the property of similarity, that $\mu(e(\gamma_X)) = t^2\mu(e(\gamma))$. Since γ is convex, by Lemma 10 we know that $\gamma_X \subset e(\gamma)$. And by the fact that elements in K are mutually disjoint, by Lemma 11, the area enclosed by them are also mutually disjoint. So

$$\mu(e(\gamma)) \geq \mu\left(\bigcup_{\gamma_X \in K} e(\gamma_X)\right) = \sum_{\gamma_X \in K} \mu(e(\gamma_X)) = t^2\mu(e(\gamma)) \sum_{\gamma_X \in K} 1 = t^2\mu(e(\gamma))|K|$$

Therefore K is finite. Thus we know that there are infinite pairs of points in γ such that their auxiliary curves have a non-empty intersection.

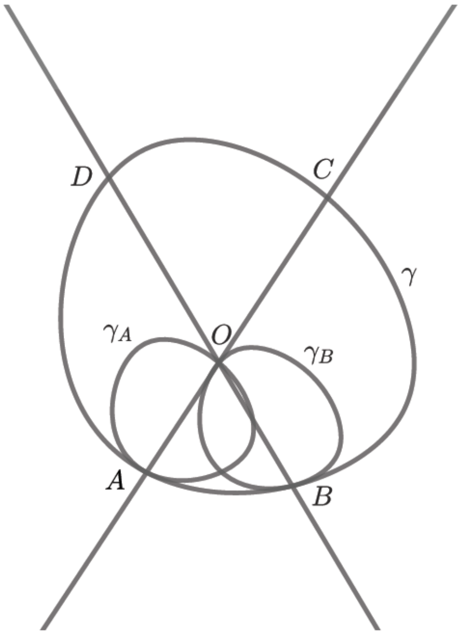

FIGURE 4. Illustration of idea

Choose any two of them, say A, B. By convexity, we can choose a point O in their intersection such that A, O, B are not collinear. [See reviewer's comment (13)] Let $C = h[A, 1/t](O), D = h[B, 1/t](O)$, consider the quadrilateral with vertices A, B, C, D, by what we have got in the beginning of this proof, $C, D \in \gamma$. Furthermore, $\triangle AOB \sim \triangle COD$ since $\angle AOB = \angle COD$ and their corresponding sides are proportional. Thus $AB//CD$ and the trapezoid $ABCD$ has a cut-ratio t by definition. Since $P \backslash K$ is infinite we could also have infinite number of such trapezoid, as desired. $\square$

3. One step upwards

From the previous section we have seen a pretty interesting result when the similarity condition is weakened a great deal. Nevertheless, we are not satisfied with keeping condition (1) in Theorem 6 only, and in Theorem 12, the properties of that infinite number of quasi-similar trapezoids are also not discussed. [See reviewer's comment (14)] In this section, we shall first derive an inscribing theorem with both conditions (1) and (2) are fulfilled, and then a corollary for the full similarity condition would follow.

3.1. Statement of our theorem

In order to state our theorem, we first define a transformation on trapezoids:

Definition 13. *We denote a plane trapezoid $ABCD$ with $AC \cap BD = O$ and $AB \leq CD$ by the 5-tuple (A, B, C, D, O). The stretching transformation is defined on the set of plane trapezoids by the mapping*

$$(A, B, C, D, O) \mapsto (A', B', C', D', O)$$

where $A', C' \in AC, B', D' \in BD$, and the cut-ratio is preserved.

It is easy to see that conditions (1) and (2) in Theorem 6 are fulfilled by any trapezoid and its stretching image. Our main theorem is hence stated as follows:

Theorem 14. *For any trapezoid T and any strictly convex C^1 Jordan curve γ, through only translation and stretching, T can be uniquely transformed to a trapezoid inscribed in γ.*

Note that an intriguing part of this theorem is that it does not need the trapezoid to be rotated, which is required in other previous pieces of work on the variations of the Trapezoidal Peg Problem. This fact gives us a number of beautiful corollaries indeed, along with the following remark:

Remark 15. *When T is rotating, the resulting inscribed trapezoid described in Theorem 14 moves continuously. [See reviewer's comment (15)]*

Note that the uniqueness condition in Theorem 14 is essential to make the statement in Remark 15 well-defined. We shall first prove Theorem 14 and Remark 15, then we will see some of their corollaries. We shall again come up with a few definitions and lemmas first.

3.2. Definitions and Lemmas

We use the same definition of cut-ratio as in the previous chapter. For convenience, we extend the last few lines of the proof of Theorem 12 to a lemma which recognize a trapezoid by the notion:

Lemma 16. *Let AC and BD be two line segment with $AC \cap BD = O$. If*

$$0 < AO : AC = BO : BD = t \leq 1/2,$$

then $ABCD$ is a trapezoid with cut-ratio t.

Proof. Since $AO : AC = t, AO = tAC = t(AO + AC)$, hence $AO : OC = t : 1 - t$, similarly $BO : OD = t : 1 - t$, it then follows that $\triangle AOB \sim \triangle COD$, which means that $AB || CD$, hence $ABCD$ is a trapezoid and its cut-ratio is t. □

Further, another quantity with respect to condition (2) is to be addressed.

Definition 17. *Let $ABCD$ be a trapezoid, and let $O = AC \cap BD$ be the intersection of its diagonals. We define the diagonal angle η of $ABCD$ as follows:*

1. *If $ABCD$ is a parallelogram, then without loss of generality we assume that $\angle AOB \leq \angle BOC$, then $\eta = \angle AOB$.*
2. *Otherwise, suppose that $AB || CD$ and $AB < CD$. Then $\eta = \angle AOB$.*

We now have the following "weakened" form of similarity between trapezoids, namely semi-similar, which is stated as follow:

Definition 18. *Let T and T' be two trapezoids. They are said to be semi-similar to each other if and only if they have the same cut-ratio and diagonal angle.*

This is the notion that we weakened similarity to, satisfying conditions (1) and (2). [See reviewer's comment (16)] Apart from trapezoid, we would also have to talk about the type of curves we are going to consider in this chapter, which we shall call it an *oval* afterwards.

Definition 19. *A Jordan curve on the plane is called an oval if*

(1) *The curve is of class C^1.*
(2) *The curve is strictly convex, i.e. any straight line could intersect it at no more than 2 points, and if there is a straight line intersecting it at only one point, then the rest of the curve lies entirely on one side of the line.*

We have several lemmas concerning this kind of plane curves.

Lemma 20. *Let γ be an oval. Choose any straight line l from the plane, then there are exactly two distinct lines that intersects γ at one point only, and is parallel to l.*

Proof. By the continuity of γ we know that there are at least two line on the plane that is parallel to l and intersects γ at one point only. It follows, since the curve is C^1, that those two lines are tangent to γ. So it remains to show that the number of such tangents is at most two. Now assume that there is a third one, say l_3, then without loss of generality the third tangent lies in between the another two. Since all three lines are parallel and distinct, the tangent points of l_1 and l_2 lies alternatively on the different sides of l_3, which violated the fact that the curve is convex. [See reviewer's comment (17)] ☐

Another two lemmas that would play a vital part of our proof is the following:

Lemma 21. *Let $ABCD$ be a parallelogram, and E, F, G, H be four points on AB, BC, CD, DA respectively such that they are not the vertices. Then for any continuous curve C_1 and C_2 such that they lie entirely in the closed region enclosed by $ABCD$ and C_1 passes through E, G and C_2 passes through F, H, $C_1 \cap C_2$ is nonempty. [See reviewer's comment (18)]*

Proof. Follows directly from the continuity of C_1 and C_2. ☐

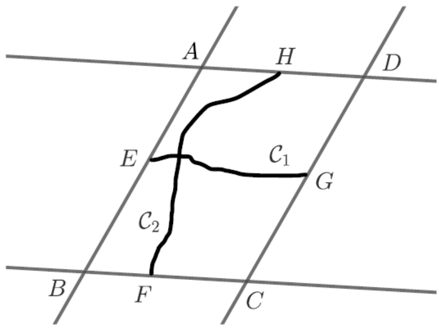

FIGURE 5. Illustration of Lemma 21

Lemma 22. *For any vector $\overrightarrow{v} \neq 0$, there exist a unique curve β on the plane with endpoints on γ in such a way that for any line $l^2 \in \mathbb{R}^2/(span(\overrightarrow{v}))$,*

1. *If l comes across one of the endpoints of β, then it is tangent to γ;*
2. *If $l \cap \gamma = \{\overrightarrow{p_1}, \overrightarrow{p_2}\}$ with $\overrightarrow{p_1} = \overrightarrow{p_2} + k\overrightarrow{v}$ for $k > 0$, then $l \cap \beta$ has only one element $\overrightarrow{p}$ such that $|\overrightarrow{p_1} - \overrightarrow{p}| = t|\overrightarrow{p_1} - \overrightarrow{p_2}|$;*
3. *Otherwise, l has no intersection with neither γ nor β*

[See reviewer's comment (19)]

Proof. We shall show the case when l is vertical and γ tangents to the y-axis at the origin with all of γ lies in quadrant I and IV. The general case is implied by means of rotation, translation and reflection.

By Lemma 20, we can choose another line l' with equation $x = d$ which is tangent to γ at some point $P = (d, d')$. The y-axis and the line l' exhibit two examples for which requirement 1 is true, and Lemma 20 again guaranteed that no other possible l can intersect γ at one point only. In addition, unless l is of the form $x = r$ with $r \in [0, d]$, it would not intersect γ at all, which case requirement 3 is true. It remains to show that any line of the form $x = r, r \in (0, d)$ intersects γ at two distinct point, and β satisfies requirement 2 with them.

The line joining the origin and P cuts the curve into two different continuous curves. Denote the part above the line by γ_1 and the another part by γ_2. It is thus clear that for any vertical line l_1 lies in between $x = 0$ and $x = d$, it must have two intersections with γ by definition, and one of them should lies on γ_1 while the another one should lies on γ_2. This could be easily seen from the continuity of γ_1 and γ_2.

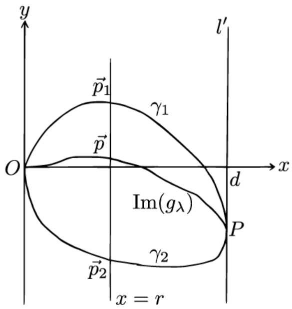

FIGURE 6. Illustation of Lemma 22

We define two functions $f_1, f_2 : [0, d] \to \mathbb{R}$ by $f_1(0) = f_2(0) = 0, f_1(d) = f_2(d) = d'$, and, $\forall r \in (0, d)$, first observe that the vertical line $x = r$ intersects γ at exactly two points since γ is convex and simple, and further from strict convexity we know that one of them must be strictly above the straight line joining O and P, and another be strictly below that. Hence one of them must lie on γ_1 and the another must lie on γ_2, so we define $f_1(r)$ to be the y-coordinate of the intersection of $x = r$ and γ_1, and $f_2(r)$ to be the y-coordinate the intersection of $x = r$ and γ_2. We now define another function, $g_\lambda : [0, d] \to \mathbb{R}^2$ by the equation

$$g_\lambda(x) = (x, \lambda f_2(x) + (1 - \lambda) f_1(x))$$

for all $x \in [0, d]$ and $\lambda \in (0, 1)$. We shall now consider the point set $\mathrm{Im}(g_\lambda)$, the image of g_λ. Firstly, since f_1, f_2 are continuous, $\mathrm{Im}(g_\lambda)$ is also a continuous curve. Therefore, take any straight line l of the form $x = r$ for some $r \in (0, d)$, it intersects γ at two different points, if we denote $\overrightarrow{p} = l \cap \mathrm{Im}(g_\lambda), \overrightarrow{p_1} = l \cap \gamma_1$ and $\overrightarrow{p_2} = l \cap \gamma_2$, since we have

$$f_1(r) - (\lambda f_2(r) + (1 - \lambda) f_1(r)) = \lambda(f_1(r) - f_2(r))$$

thus $|\overrightarrow{p_1} - \overrightarrow{p}| = \lambda|\overrightarrow{p_1} - \overrightarrow{p_2}|$. This, along with the fact that the curve is simple, further tell us that the only intersections of it with λ are its endpoints, i.e. $g(0)$ and $g(d)$.

Take $\beta = \mathrm{Im}(g_t)$ yields our desired curve. [See reviewer's comment (20)] $\square$

We denote the continuous curve in the above lemma by $\beta(\overrightarrow{v})$.

3.3. Proof of Theorem 14

Here we shall prove Theorem 14. Note that the theorem actually consists of two parts: the existence of such an inscribed trapezoid, and the uniqueness of it i.e. If there are two such trapezoids, then they coincide, which could then help us formulate the statement in Remark 15. [See reviewer's comment (21)] We shall tackle them one by one.

3.3.1. Existence

Let γ be an oval and let $ABCD$ be a trapezoid, where $AB||CD$, with cut-ratio t and diagonal angle η.

We know from our assumption that γ is differentiable, i.e. it has a unique tangent at each point, that for any two linearly independent vector $\overrightarrow{c_1}, \overrightarrow{c_2}$, any of the endpoints of $\beta(\overrightarrow{c_1})$ and $\beta(\overrightarrow{c_2})$ are distinct.

Consider the vector $\overrightarrow{V} = \overrightarrow{CA}$. Now rotate the vector V clockwisely by angle η to get another vector $\overrightarrow{W}$, which equals $b\overrightarrow{DB}$ for some $b > 0$.

Recall that $0 < \eta < \pi$, which means that $\overrightarrow{V}$ and $\overrightarrow{W}$ are linearly independent, thus again by the properties we found above, they have no intersection on γ. Put it in another way, if they have an intersection, they it ought to lie in the interior of the area enclosed by γ.

The rest is to show the existence of such an intersection. It can be seen that the endpoints of these two curves lies on the two pairs of tangents to C which are parallel to $\overrightarrow{V}, \overrightarrow{W}$, respectively, which intersects to form a parallelogram. Since γ is convex, the parallelogram must inscribe γ. Also, since the curves, by their definitions, lie entirely in the closed region enclosed by γ, they lie entirely in the closure of the parallelogram.

Again due to the fact that all four intersection points are distinct, Lemma 21 implies the existence of an intersection, which, as said above, ought to lie in the interior of the area enclosed by γ.

Hence the existence of the desired intersection is proved. Choose one of the intersections, denote it by $\overrightarrow{s}$. We then consider the lines $l_{\overrightarrow{V}} = \overrightarrow{s} + \mathrm{span}(\overrightarrow{V})$ and $l_{\overrightarrow{W}} = \overrightarrow{s} + \mathrm{span}(\overrightarrow{W})$. Then obviously

$$\gamma \cap (l_{\overrightarrow{V}} \cup l_{\overrightarrow{W}}) = (\gamma \cap l_{\overrightarrow{V}}) \cup (\gamma \cap l_{\overrightarrow{W}})$$

have four elements which are the vertices of a trapezoid with cut-ratio t and diagonal angle η.

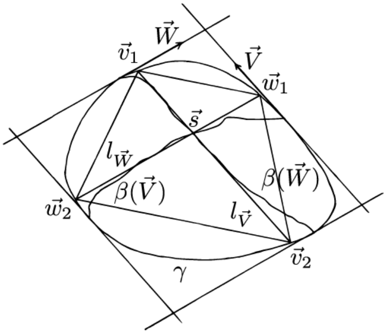

FIGURE 7. The inscribed trapezoid

Suppose $\gamma \cap l_{\overrightarrow{V}} = \{\overrightarrow{v_1}, \overrightarrow{v_2}\}$ with $\overrightarrow{v_1} = \overrightarrow{v_2} + k_v\overrightarrow{V}, k_v > 0$ and $\gamma \cap l_{\overrightarrow{W}} = \{\overrightarrow{w_1}, \overrightarrow{w_2}\}$ with $\overrightarrow{w_1} = \overrightarrow{w_2} + k_w\overrightarrow{W}, k_w > 0$, the desired translation would be the one sending O to $\overrightarrow{s}$, which sends (A, B, C, D, O) to $(A', B', C', D', \overrightarrow{s})$ for some points A', B', C', D', and the stretching would send $(A', B', C', D', \overrightarrow{s})$ to $(\overrightarrow{v_1}, \overrightarrow{w_1}, \overrightarrow{v_2}, \overrightarrow{w_2}, \overrightarrow{s})$, which is inscribed in γ.

Obviously this is an appropriate construction, and the existence is proved.

3.3.2. Uniqueness

In this subsection, we shall show the uniqueness of the intersection of $\beta(\overrightarrow{V})$ and $\beta(\overrightarrow{W})$. By our last construction of trapezoids in the previous section, we know that if there are two distinct ones, then there must be two distinct trapezoids inscribed in the same oval, namely $ABCD$ and $A'B'C'D'$, such that

$$AB \leq CD, AB||CD, A'B' \leq C'D', A'B'||C'D', AC||A'C', BD||B'D',$$

and, if we let $O = AC \cap BD, O' = A'C \cap B'D'$,

$$\frac{AO}{OC} = \frac{BO}{OD} = \frac{A'O'}{O'C'} = \frac{B'O'}{O'D'} = \frac{t}{1-t} < 1$$

since $t \leq 1/2$. We may assume as well that no three of the vertices of these two trapezoids are collinear, which is obvious.

The case in which the two said trapezoids are congruent is trivial. Now assume that they are not, then there must be one pair of parallel diagonals which have different length. With a suitable rotation and reflection, the shorter one is on the right-hand side on the longer one, and the two diagonals are vertical. Without loss of generality $AC > A'C'$ satisfy our requirement.

We shall make use of the following facts in our later proof:

Lemma 23. *Any convex curve cannot inscribe a concave polygon.*

Proof. Assume for sake of contradiction that a convex curve inscribes the concave polygon. We consider the convex hull of that polygon. By concavity there is at least a vertex of that polygon which is not in the boundary of its convex hull, but this would mean that it is in the interior of the convex curve, which means that the vertex is not on the curve, a contradiction. □

Lemma 24. *Let $ABCD, A'B'C'D'$ and O, O' be as defined above. Then the lines AA', CC', OO' are concurrent, also the lines BB', DD', OO' are either concurrent or pairwise parallel.*

Proof. We shall show the lemma more generally: For any two parallel line segments $AC, A'C'$ and the points O, O' on them respectively with $AO : OC = A'O' : O'C'$, the straight lines AA', OO', CC' are either concurrent or pairwise parallel.

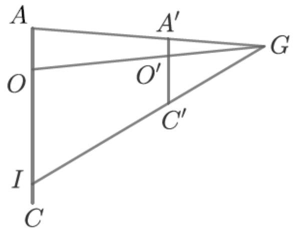

Figure 8. Illustation of Lemma 24

Clearly they will be parallel if and only if $AO = A'O'$, hence we assume that they are not. Let G be the intersection point of AA' and OO', And let I be the intersection of GC' and AC. $\triangle GO'C' \sim \triangle GOI$ and $\triangle GO'A' \sim \triangle GOA$ by AAA, therefore

$$OI = \frac{GO}{GO'}O'C' = \frac{AO}{A'O'}O'C' = AO\frac{OC}{AO} = OC$$

Hence $I = C$. □

Now we begin our proof to the contrapositive of our assertion.

Proposition 25. *If such $ABCD$ and $A'B'C'D'$ exist on the plane, then among these eight points, there must be at least one set of points forming a concave polygon.*

Proof. We first divide the plane by the positions of A, C, A', C' into eight regions, as shown in Figure 9. Our proof will base on the casework through these eight regions.

For convenience, we denote the slope of a straight line l by slope_l. Firstly, if any one of the remaining four points, B, D, B', D', in region I, III, or VII, then we could find a concave quadrilateral by joining that points with three points from $\{A, A', C, C'\}$ except the one with the greatest distance from it. And if any one of the four points is in region V, then joining it with A, A', C, C' will yield a concave pentagon. Thus, it remains for us to deal with region VI, IV, VIII and II.

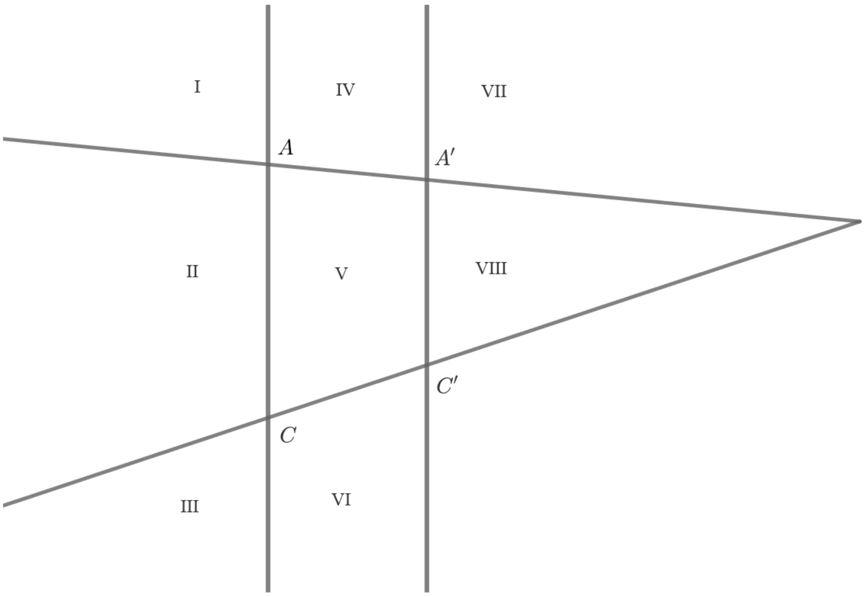

FIGURE 9. Plane division

1. Case $B' \in \text{VI}$

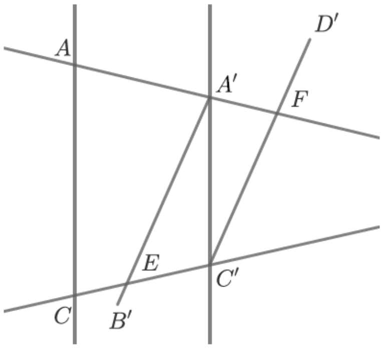

FIGURE 10. Case $B' \in \text{VI}$

Let E be the intersection of line segment $A'B'$ and CC', which exists since $B' \in$ VI, and let F be the intersection of the rays AA' and $C'D'$. Now D' is on ray $C'F$, but we clearly have

$$C'F < A'E < A'B'$$

since $A'B' < C'D'$, D' is not on the line segment $C'F$, therefore it is in region VII, which case have been dealt with.

2. Case $B' \in$ IV

   Clearly,

$$\text{slope}_{C'D'} = \text{slope}_{A'B'} \leq \text{slope}_{AA'} < \text{slope}_{CC'}$$

Therefore the ray $C'D' \subset VII$, thus $D' \in$ VII, which is an established case.

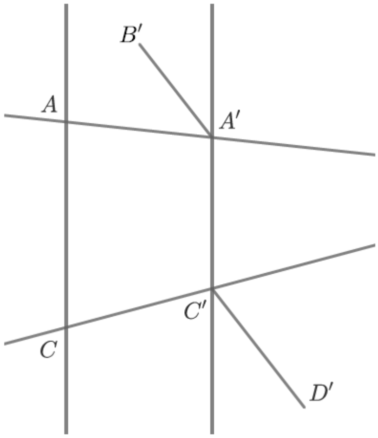

FIGURE 11. Case $B' \in$ IV

3. Case $B' \in$ II

   We argue by consider the length ratio of BD and $B'D'$.

   If $BD \leq B'D'$, clearly the intersection of OO' and BB' is in region II, By Lemma 24, DD' and OO' also intersect in region II. We denote this intersection by H.

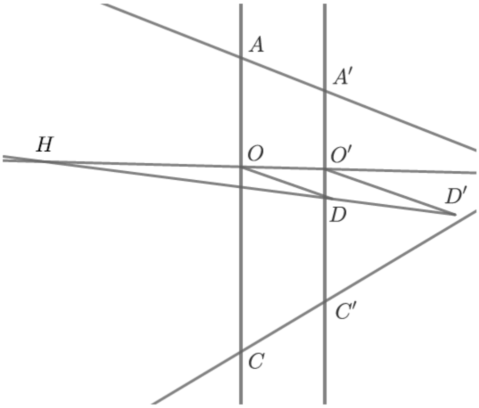

FIGURE 12. Case $B' \in$ II

Now, since D' is on the right-hand side of $A'C'$, either $D' \in$ VII, which case we have proved, or $D' \in$ VIII, then the fact that $H \in$ II would guarantee the intersection of the line $D'D$ and line segment AC, hence D is in the interior of the triangle ACD', therefore $AD'DC$ is concave.

The case is similar if $BD < B'D'$. Again, we consider the position of H. Obviously H lies on the ray OO', so either $H \in OG$ where $G = AA' \cap CC' \cap OO'$ (exists by Lemma 24), or H lies in the part of OO' on the right-hand side of G. The former guarantees us that $BB' \cap AC \neq \emptyset$ and thus that B' is in the interior of $\triangle BAC$, giving a concave quadrilateral $BAB'C$.

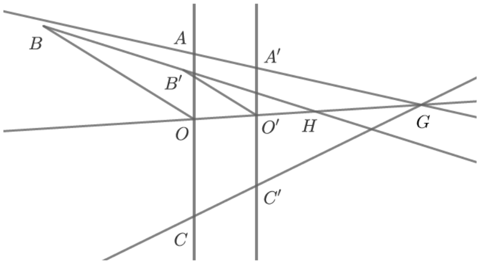

FIGURE 13. Case $B' \in$ II

And for the latter, it is obvious that we have

$$\text{slope}_{CC'} < \text{slope}_{DD'} < \text{slope}_{AA'},$$

therefore DD', again, intersects AC, so $D'AC$ contains D, which leaves a concave $D'DAC$.

4. Case $B' \in$ VIII

The case $BD \leq B'D'$ is an analogy to the case that $B' \in$ II by relabeling B, B' as D, D' and vice versa. If $BD > B'D'$, the case is analogous again if

$D' \notin \mathrm{IV} \cup \mathrm{VI}$. But if $D' \in \mathrm{IV} \cup \mathrm{VI}$, we shall argue by consider the slope of $D'H$. If either $\mathrm{slope}_{AA'} \leq \mathrm{slope}_{D'H}$ or $\mathrm{slope}_{CC'} \geq \mathrm{slope}_{D'H}$, then it would mean that $D \in \mathrm{I} \cup \mathrm{III}$, which case we have done.

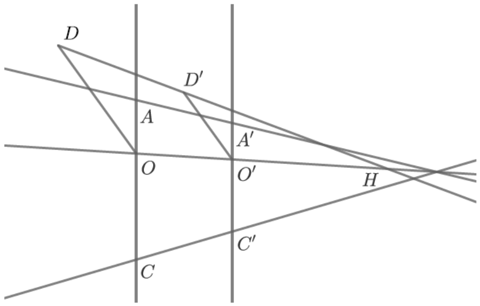

FIGURE 14. Case $B' \in \mathrm{VIII}$

Otherwise it is clear that H is on the right-hand side of $A'C'$, and that

$$\mathrm{slope}_{AA'} > \mathrm{slope}_{BB'} > \mathrm{slope}_{CC'}$$

This means that $BB' \cap AC \neq \emptyset$, therefore $ABCB'$ is a concave quadrilateral.

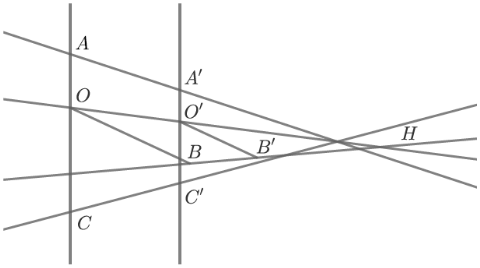

FIGURE 15. Case $B' \in \mathrm{VIII}$

Thus, wherever B' is, there must be a concave polygon formed with the eight vertices of $ABCD$ and $A'B'C'D'$, and the proposition is proved.

Thus the uniqueness is proved. With this, observe that when $ABCD$ rotates, $\beta(\overrightarrow{V})$ and $\beta(\overrightarrow{W})$ moves continuously, which immediately give us the statement of Remark 15 along with the fact γ is continuous.

$\square$

3.4. Corollaries of the theorem

Obviously, since there are uncountable number of directions, by rotating $ABCD$, we have

Corollary 26. *An oval can inscribe uncountably many trapezoids that is semi-similar to any prescribed trapezoid.*

And also, observe that the only place where we need differentiability is the need of unique tangent. But we know that a convex curve is differentiable almost everywhere, that is, it has unique tangent parallel to our given direction with probability 1, thus,

Corollary 27. *Theorem 14 is true for general strictly convex curves with probability 1.*

A more intriguing result which we can obtain together with the statement in Remark 15 is the implication for full similarity. Obviously what we want must come from playing with condition (3) given in Theorem 6, thus it is not surprising to have the following definition.

Definition 28. *A trapezoid $ABCD$, named clockwisely with $AB||CD$ and $AC \cap BD = O$, is said to be with diagonal ratio δ, if one of the following conditions are satisfied*

1. *The cut-ratio of the trapezoid is not* $1/2$, *and* $AC : BD = \delta$.
2. *The cut-ratio is* $1/2$, *in which case $ABCD$ is a parallelogram, and $AC : BD = \delta$ given that $\angle AOB \leq \angle BOC$.*

Corollary 29. *Let T be a trapezoid with diagonal ratio δ, γ any oval. T can be inscribed in γ up to similarity if and only if γ inscribes two other trapezoids T_1 (with diagonal ratio δ_1) and T_2 (with diagonal ratio δ_2) that are semi-similar to T, and*

$$\delta_1 \leq \delta \leq \delta_2$$

[See reviewer's comment (22)]

Proof. For necessity, just take $T_1 = T_2 = T$. And for sufficiency, by Theorem 14 and Remark 15, we can rotate T by angle $\theta \in [0, 2\pi)$ to get a continuously moving set of inscribed trapezoids T_θ. Define the function $f : [0, 2\pi) \to \mathbb{R}_+$ by assigning $\theta \mapsto \delta_\theta$ where δ_θ is the diagonal ratio of T_θ. The result then follows from the intermediate value theorem. □

Remark 30. *It is possible for a trapezoid to be with multiple diagonal ratios, specifically when the trapezoid is actually a rhombus. This fact, however, could be used to prove that every oval can inscribe a square. This special occasion is the work done by* [3].

4. Conclusion

4.1. Summary

As we have demonstrated in Theorem 14 and Corollary 29, through decomposing the similarity condition into different components, and analyzing the family of inscribed trapezoids up to a weakened condition, we then obtain a result that illustrates properties which lead to the answer to whether the given trapezoid can be inscribed in our class of curves. Specifically, we first demonstrated how we decompose the similarity condition among trapezoids into three separate conditions, and that a weakened similarity condition could yield an infinite family of inscribed trapezoids in Theorem 12. Afterwards, we investigated the properties of the infinite family of inscribed trapezoids when we require only semi-similarity, which is one of the weakened form of similarity, and this yield Theorem 14. This theorem proved the existence and uniqueness of an inscribed trapezoid semi-similar to a prescribed trapezoid in a curve, via only translation and stretching. And along with Remark 15, we answered Problem 4 within the type of curves we called 'ovals', which is the result shown in Corollary 29.

4.2. Possible directions for future research

This paper merely focuses on convex curves with good enough properties. However, the methodology of decomposing similarity might also be applicable to inscribing problems of trapezoids in other classes of Jordan curves. In particular, note that the proof of uniqueness of Theorem 14 only relies on the strict convexity of our desired curve. If we could extend the existence proof as well as Remark 15 to strictly convex curves that are not necessarily continuously differentiable, then it is possible to find an analogy to the important Corollary 29 among these more general type of curves.

Furthermore, higher-dimensional perspectives of the problem is also an interesting direction. [See reviewer's comment (23)] Though it seems hard to directly extend the problem to arbitrary dimensions, one could start by considering how can trapezoids (or general notions of it) be inscribed in embeddings $\gamma : \mathbb{S}^1 \to \mathbb{R}^3$, i.e. knots, with specific properties.

In addition, the result we have obtained in this paper is also possible to be simplified. Although we have obtained such an equivalent condition, its formulation still complex and should have been more inspiring. However, one should be reminded that it is not a simple work to do, since one would need to translate the characteristics of a trapezoid which is often simple, to some specific properties of a curve, whose description is often complex.

REFERENCES

[1] A. Akopyan and S. Avvakumov, *Any cyclic quadrilateral can be inscribed in any closed convex smooth curve*, ArXiv e-prints, December 2017. URL https://arxiv.org/abs/1712.10205.

[2] A. Emch, *On the medians of a closed convex polygon*, American Journal of Mathematics, 37(1):19-28, 1915.

[3] A. Emch, *Some properties of closed convex curves in a plane*, American Journal of Mathematics, 35 (4): 407-412, 1913.

[4] A. Emch, *On some properties of the medians of closed continuous curves formed by analytic arcs*, American Journal of Mathematics, 38(1): 6-18, 1916.

[5] R. Fenn, *The table theorem*, Bulletin of the London Mathematical Society, 2(1): 73-76, 1970. doi: 10.1112/blms/2.1.73.

[6] V. V. Makeev, *On the quadrangles inscribed in a closed curve*, Mathematical Notes, 57(1-2), 1995

[7] B. Matschke, *Equivariant topology methods in discrete geometry*, PhD thesis, Freie Universitat Berlin, 2011.

[8] B. Matschke, *A survey on the square peg problem*, Notices of the American Mathematical Society, 2014.

[9] I. Pak, *Lectures on discrete and polyhedral geometry*, Lecture Notes at UCLA, 2010. URL http://www.math.ucla.edu/∼pak/geompol8.pdf.

[10] T. Tao, *An integration approach to the Toeplitz square peg problem*, ArXiv eprints, November 2016. URL https://arxiv.org/abs/1611.07441v2.

[11] L. G. Schnirelman, *On some geometric properties of closed curves*, Uspekhi Matematicheskikh Nauk, 10: 34-44, 1944.

[12] W. R. Stromquist, *Inscribed squares and square-like quadrilaterals in closed curves*, Mathematika, 36: 187-197, 1989.

[13] O. Toeplitz, *Ueber einige aufgaben der analysis situs*, Verhandlungen der Schweizerischen Naturforschenden Gesellschaft in Solothurn, 4(197): 29-30, 1911.

Reviewer's Comments

The Square Peg Problem, proposed by Toeplitz about a century ago, asks whether every Jordan curve (i.e. a continuous simple closed curve on the plane) inscribes a square. A famous problem of significance in topology and geometry, the Square Peg Problem has since been open for general Jordan curves and elicited numerous proofs of existence of square pegs for special classes of curves, with Terry Tao's proof being the latest development of the problem. Inspired by the Square Peg Problem, the paper under review addresses one of its variant, namely the Trapezoidal Peg Problem, which asks whether, for any given trapezoid T, a Jordan curve inscribes a trapezoid which is similar to T in a weak sense. The author starts with a characterization of similarity of two trapezoids using three conditions involving the angle between the diagonals and ratios of lengths of some line segments of the trapezoids. He then defines two weak notions of similarity of trapezoids by removing one or two conditions from the characterization of similarity and presents two results affirming the existence of trapezoidal peg weakly similar to any given T for some special classes of Jordan curves.

1. (Theorem 12) Two trapezoids are said to be quasi-similar if the intersection points of their diagonals divide their respective diagonals in the same ratio. Then for any trapezoid T and any *convex* Jordan curve γ, there exist infinitely many trapezoids which are inscribed by γ and quasi-similar to T.
2. (Theorem 14) A trapezoid is said to be obtained from another by *stretching* if the two trapezoids are quasi-similar and the opposite vertices of the trapezoids are collinear. For any trapezoid T and any *strictly convex* C^1 Jordan curve γ called *oval*, there exists a unique trapezoid which is inscribed by γ and obtained from T through translation and stretching only (rotation is not necessary).

From Theorem 14, the author deduces Corollary 29, which gives, for any oval γ, a necessary and sufficient condition for a given trapezoid T to be inscribed by γ up to similarity, thereby giving at least a partial solution to the Trapezoidal Peg Problem. In proving these main results, the author makes crucial use of intermediate value theorem to pinpoint the intersection of the diagonals of the desired trapezoid inscribed in an oval, as well as to show the existence of the trapezoid as in Corollary 29. Other tools the author employs include homothety of convexity Jordan curves and their convexity for Theorem 12, and elementary geometry of triangles in the uniqueness part of Theorem 14.

In general, the paper is so well-structured that it exceeds the expectation from a high school student. The author gives a very nice introduction to the history of and research developments surrounding the square peg problem, which serves to motivate the trapezoidal peg problem he attempts to address. He also explains the differences between the square and trapezoidal peg problem in order to justify his attempt to address the latter problem. He develops the main results in painstaking details in two stages, starting with Theorem 12, whose proof is easier due to the weakest condition of similarity assumed, and then progressing to Theorem 14, whose proof occupies a major part of the paper and involves some case-by-case analysis.

Nevertheless, this paper is strewn with grammatical mistakes and typos, especially in long, compound sentences. Wrong use of commas is common in long sentences where full stops should have been used instead to break up the sentences. Some statements can be formulated more clearly. For instance, Lemma 22 does not read well and can be better rephrased. The following are mistakes the reviewer found in the paper and suggestions for improvement.

1. The reviewer has comments on the wordings, which have been amended in this paper.
2. it is better to rewrite the sentence as 'even it is already more than one hundred years since Conjecture 1.1 was stated,'
3. should be rewritten as '...problem *through* the years *according* to a recent survey, *from* which selected ones are listed...'.
4. 'focused' should be 'focuses'. The latter part of the sentence should rewritten as '...be able to be inscribed...'.
5. '...is still conjectured' should be rewritten as '...is still in conjectural status'.
6. 'as said above' can be rewritten as 'as *mentioned* above'.
7. '...it have not finished being reviewed' can be rewritten as 'it is still under review'.
8. 'be justified' should be 'justified'. '...types of curves can some...' should be '...types of curves which can inscribe some...'. Delete 'be inscribed, in which it can' and replace it with 'As such we formulate the following problem'.
9. The sentence 'And our idea...actually consists of' does not read well. A suggested correction is 'Our idea is to look into the conditions similarity between trapezoids actually consist of'.
10. The reviewer believes by 'area' the author actually means 'the set of points'.
11. the auxilliary curve γ_X actually also depends on the homothety factor t, so it is better to use the notation $\gamma_{X,t}$ instead to stress its dependence on t.
12. The reviewer believes the author actually means 'such that each of them as no intersection...'.
13. it is not clear why on can choose a point O such that A, O and B are not collinear. The author should explain.
14. 'the properties of that infinite number of quasi-similar trapezoids are also....' should read 'the properties that there are infinitely many quasi-similar trapezoids inscribed in a convex Jordan curve are also...'.
15. the author should specify how T rotates. For instance, about which point does T rotate?
16. The sentence can be rewritten as 'This is the notion to which we weaken similarity, and which satisfies conditions...'.
17. The reviewer believes the author actually means the curve is *strictly* convex.

18. The reviewer thinks by 'the closed region enclosed by $ABCD$' the author actually means the interior enclosed by $ABCD$.
19. the statement is not formulated clearly. For one, it does not specify whether t is a fixed constant or arbitrary. One suggested correction: For any vector $\vec{v} \neq \vec{0}$ and any line $\ell \in \mathbb{R}^2 \setminus (\mathrm{span}(\vec{v}))$ and a fixed constant t, there exists a unique curve β on the plane...
20. in p.12, line -11, λ is used to denote the scaling factor, but then in p.13, line 1, t is used to mean the same thing. The author should have consistently used the same notation to denote the same thing. Besides, the curve $\beta(\vec{v})$ should better be denoted by $\beta_\lambda(\vec{v})$ or $\beta_t(\vec{v})$ to stress its dependence on the scaling factor.
21. The reviewer believes that by 'they', the author means the curves $\beta(\vec{V})$ and $\beta(\vec{W})$. He should have clearly indicated what 'they' refers to.
22. in fact this corollary can be further strengthened by letting $\delta_1 = \min f$ and $\delta_2 = \max f$. δ_1 and δ_2 exist because of the continuity of f and compactness of the circle (in fact f can be regarded as a function on a circle).
23. The reviewer believes by 'perspectives' the author actually means 'analogues'.