Hang Lung Mathematics Awards
Vol. 6 (2014)

Hang Lung Mathematics Awards 2014

Honorable Mention

Classification of Prime Numbers by Prime Number Trees

Team Members: Man Him Ho, Chun Lai Yip, Yat Wong, Yin Kei Tam
Teacher: Mr. Alexander Kin Chit O
School: G.T. (Ellen Yeung) College

Hang Lung Mathematics Awards
Vol. 6 (2014), pp. 133–150

CLASSIFICATION OF PRIME NUMBERS BY PRIME NUMBER TREES

TEAM MEMBERS
Man Him Ho, Chun Lai Yip,
Yat Wong, Yin Kei Tam

TEACHER
Mr. Alexander Kin Chit O

SCHOOL
G.T. (Ellen Yeung) College

Abstract. The traditional sieve of Eratosthenes gives a simple algorithm for finding all prime numbers. However, prime numbers seem appear unpredictably but with regular population ratio in the ranges of integers, as Gauss found a density function of prime numbers within a range of x. On the other hand, there are a few methods of classification of prime numbers. We developed a new classification of prime numbers by prime number trees. In the prime number trees, the followed number is generated by attaching a digit either 1, 3, 7, or 9 to the right hand side of the preceding prime number. If the number generated remains prime, then the process is continued, otherwise it is stopped. The prime number trees group prime numbers with similar digits together and show the elegancy of a shorthand of prime numbers. This method also shows a regular classification of prime numbers.

Part A. Prime Number Trees in the Decimal Number System

1. Introduction

1.1. Prime numbers

The traditional sieve of Eratosthenes gives a simple algorithm for finding all prime numbers. However, prime numbers seem appear unpredictably but with regular population ratio in the ranges of integers, as Gauss found the density function of prime numbers within a range of x to be $\approx \dfrac{1}{\ln x}$ [2]. On the other hand, there are a few methods of classification of prime numbers. This is probably due to the lack of understanding the properties of prime numbers. No formulae has been found to compute prime numbers successfully. For example, Fermat numbers, Mersenne numbers are not all prime. Fermat Little Theorem only gives the property of prime numbers but not a method of inspecting a number to be prime.

1.2. Classification of Prime Numbers

According to Erdős -Selfridge's classification of prime numbers [1], there are infinite classes of prime numbers with each class containing infinite prime numbers. In each class, there is not a clear regulation of appearing of prime numbers. This is shown in Chapter 2.

We developed a new classification of prime numbers by prime number trees. In the prime number trees, the followed number is generated by attaching a digit either 1, 3, 7, or 9 to the right hand side of the preceding prime number. If the number generated remains prime, then the process is continued, otherwise it is stopped. For example, the prime number tree of 2 contains 2, 23, 233, 2333, 23333, which are all prime. The prime number trees group prime numbers with similar digits together and show the elegancy of a shorthand of prime numbers. This method also shows a regular classification of prime numbers.

2. Traditional Classification of Prime Numbers

2.1. Erdős -Selfridge's Classification of Prime Numbers

Paul Erdős and John Selfridge classified prime numbers as follows. For p to be a prime number,

(i) if the largest prime factor of $p+1$ is 2 or 3, then p is in Class 1,

(ii) if the largest prime factor of $p+1$ is in class c, then p is in Class $(c+1)$.

2.2. Classes of Prime Numbers

Class 1: 2, 3, 5, 7, 11, 17, 23, 31, 47, 53, 71, 107, 127, 191, 383, . . .

e.g. For the prime number $p=3$, $p+1=4=2\times 2$, the largest prime factor of $(p+1)$ is 2. Then $p=3$ is in Class 1.

For the prime number $p=5$, $p+1=6=2\times 3$, the largest prime factor of $(p+1)$ is 3. Then $p=5$ is in Class 1.

Prime numbers in Class 1 are of the form $2^i 3^j - 1$ for $i\geq 0$ and $j\geq 0$.

Class 2: 13, 19, 29, 41, 43, 59, 61, 67, 79, 83, 89, 97, 101, 109, 131, 137, 139, . . .

e.g. For the prime number $p = 13$, $p + 1 = 14 = 2 \times 7$, the largest prime factor of $(p + 1)$ is 7, which is in Class 1. Then $p = 13$ is in Class 2.

Class 3: 37, 103, 113, 151, 157, 163, 173, 181, 193, 227, 233, 257, 277, 311, 331, 337, . . .

e.g. For the prime number $p = 37$, $p + 1 = 38 = 2 \times 19$, the largest prime factor of $(p + 1)$ is 19, which is in Class 2. Then $p = 37$ is in Class 3.

Class 4: 73, 313, 443, 617, 661, 673, 677, 691, 739, 757, 823, 887, 907, 941 . . .

e.g. For the prime number $p = 73$, $p + 1 = 74 = 2 \times 37$, the largest prime factor of $(p + 1)$ is 37, which is in Class 3. Then $p = 37$ is in Class 4.

Therefore, the Erdős-Selfridge classification of primes classifies prime numbers according to their neighbours.

3. Classification of Prime Numbers by Prime Number Trees

3.1. Prime Number Trees

The *Prime Number Tree* is a sequence or a family of prime numbers. It starts from a prime number and ends again with a prime number. In a prime number tree, the followed number is generated by attaching 1, 3, 7 or 9 to the right hand side of the preceding prime number and it remains prime. The sequence ends when no more prime number can be generated by doing so. e.g. 2, 23, 233, 2333, 23333 are all prime. However, 233331, 233333, 233337 and 233339 are no longer prime. The prime number tree of 2 is shown below.

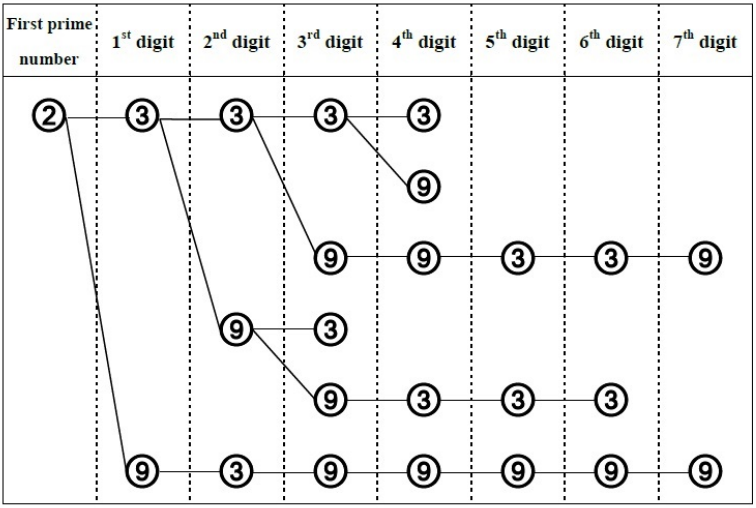

Table 1. Prime Number Tree of 2. The prime number tree ends when no more prime number can be generated.

Definition 1. Scheme of Odd Unit Integers Plus Multiple of Ten (SOUIPMT) *If a new prime number is produced by multiplying an old prime number by 10 and followed by an addition of either 1, 3, 7 or 9, this new prime number is said to be generated by the Scheme of Odd Unit Integers Plus Multiple of Ten (SOUIPMT).*

e.g. $p_1 = 2$ is prime, $p_2 = 10 \times 2 + 3 = 23$ is also prime. p_2 is said to be generated by SOUIPMT.

Definition 2. Fundamental Prime Number *A fundamental prime number is a prime number which cannot be generated by SOUIPMT.*

e.g. The prime number $23 = 10 \times 2 + 3$. It can be generated by SOUIPMT. Therefore, 23 is not a fundamental prime number.

e.g. The prime number $89 = 10 \times 8 + 9$. Since 8 is not prime, 89 cannot be generated by SOUIPMT. Therefore, 89 is a fundamental prime number.

Definition 3. Tail Prime Number *A tail prime number is a prime number which cannot generate a new prime number by SOUIPMT.*

e.g. 23333 is prime. However, 233331, 233333, 233337, 233339 are all composite. Therefore, 23333 is a tail prime number.

Definition 4. Prime Number Tree *A prime number tree is composed of prime number sequences which start from the same fundamental prime number and the followed prime numbers generated by SOUIPMT, one by one. The sequences end with different tail prime numbers so that there are many branches in it.*

The prime number trees of 3, 5 and 7 are shown below.

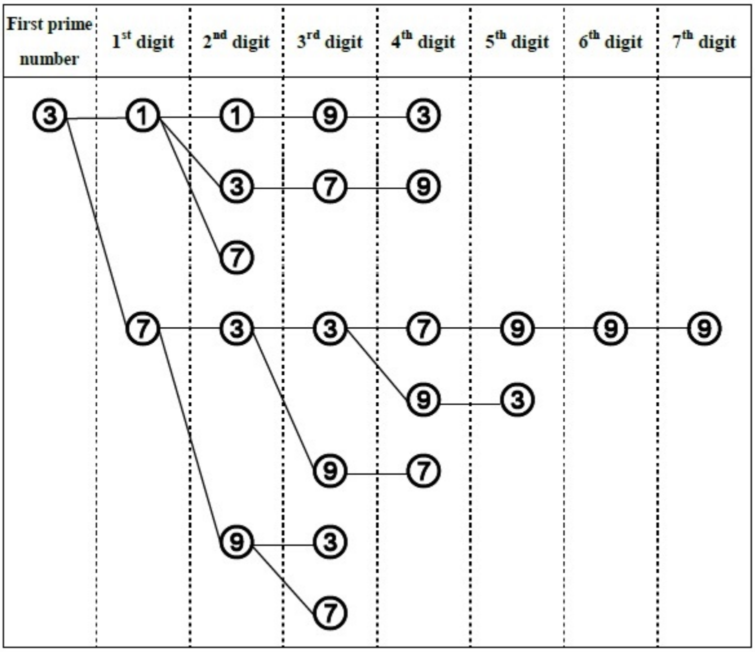

Table 2. Prime Number Tree of 3

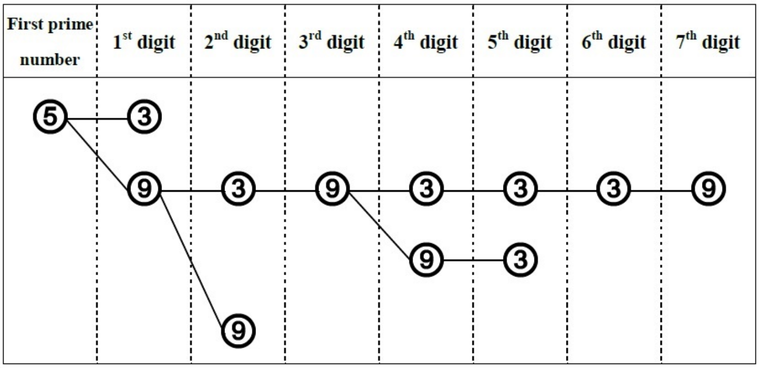

Table 3. Prime Number Tree of 5

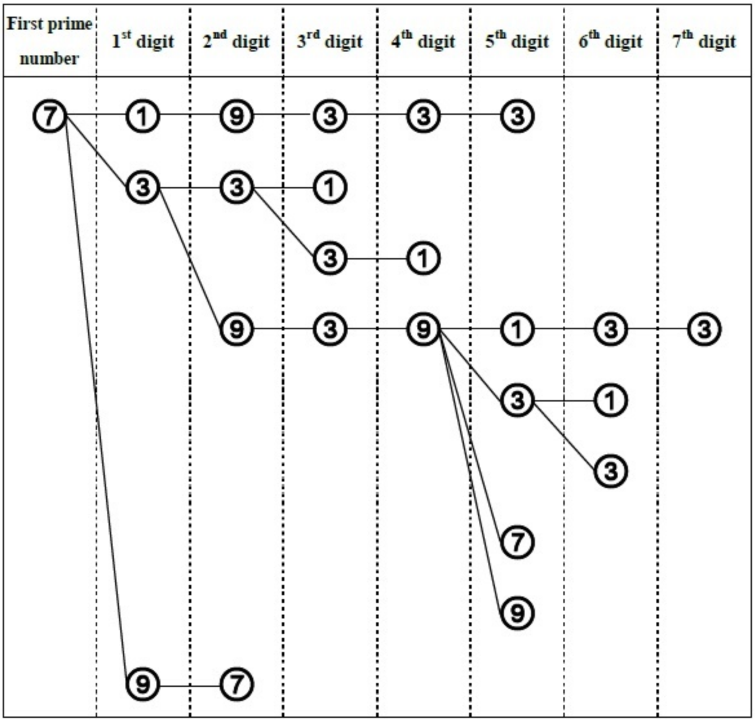

Table 4. Prime Number Tree of 7

Definition 5. Alone Prime Number *An alone prime number is a prime number tree containing only the fundamental prime number. The fundamental prime number is also the tail prime number.*

A list of alone prime numbers is shown in Table 5 below.

89	**389**
107	**443**
167	**457**
251	**461**

Table 5. Alone prime numbers

3.2. Properties of Tail Prime Numbers

For tail prime numbers or alone prime numbers p, particularly with $p \equiv 2 \pmod 3$, we can construct four composite numbers as $q_1 = 10p+1, q_2 = 10p+3, q_3 = 10p+7$ and $q_4 = 10p+9$ such that

$$\begin{cases} 10p+1 = p_1 \times n_1, \\ 10p+3 = p_2 \times n_2, \\ 10p+7 = p_3 \times n_3, \\ 10p+9 = p_4 \times n_4, \end{cases}$$

where p_1, p_2, p_3 and p_4 are the smallest odd prime factors whereas n_1, n_2, n_3 and n_4 are odd positive integers greater than 1. It is found that $q_3 - q_1 = 6$ and

$$p_3 \times n_3 - p_1 \times n_1 = 6, \tag{1}$$

which is an indefinite linear Diophantine equation with conditions $10p+1 = p_1 \times n_1$ and $10p+7 = p_3 \times n_3$. The solution to the equation (1) can be found as $p_3 = p_1 = 3$ and $n_3 - n_1 = 2$. Hence $q_1 = 10p+1 = 3n_1$, which is a multiple of 3.

In this case there are two remarks.

1. For p to be a tail prime number and $p \equiv 1 \pmod 3$, it does not imply the solution to equation (1) is $p_3 = p_1 = 3$ and $n_3 - n_1 = 2$. The counter-example is the tail prime number $p = 3793$ with $37931 \equiv 2 \pmod 3$, $37937 \equiv 2 \pmod 3$, $37933 \equiv 0 \pmod 7$ and $37939 \equiv 0 \pmod{11}$.
2. If the results $p_3 = p_1 = 3$ and $n_3 - n_1 = 2$ are given and with the same condition, it does not imply p must be prime. $q = 2009$ is a counter-example. $20091, 20093, 20097$ and 20099 are all composite and $20091 \equiv 0 \pmod 3$, $20097 \equiv 0 \pmod 3$. However, 2009 is not prime.

Theorem 6. Theorem of Tail Prime Number *Let p be a tail prime number with $p \equiv 2 \pmod 3$. Then $q_1 = 10p+1$, $q_2 = 10p+3$, $q_3 = 10p+7$ and $q_4 = 10p+9$ all are odd composite numbers by the definition of tail prime number. Let*

$$\begin{cases} 10p+1 = p_1 \times n_1, \\ 10p+3 = p_2 \times n_2, \\ 10p+7 = p_3 \times n_3, \\ 10p+9 = p_4 \times n_4, \end{cases}$$

where p_1, p_2, p_3 and p_4 are the smallest odd prime factors whereas n_1, n_2, n_3 and n_4 are odd positive integers. Then $q_3 - q_1 = 6 = p_3 \times n_3 - p_1 \times n_1$. The solution to the equation is if and only if $p_3 = p_1 = 3$ and $n_3 - n_1 = 2$.

[See reviewer's comment (2)]

Proof. We prove "if" first [See reviewer's comment (3)]. For $p \equiv 2 \pmod 3$ and $10p+1 = p_1 \times n_1$, $p_1 \times n_1 \equiv 10p+1 \pmod 3$. Since $10p+1 \equiv 1 \cdot p + 1 \equiv 1 \cdot 2 + 1 \equiv 0 \pmod 3$, $p_1 \times n_1 \equiv 0 \pmod 3$. It is either $p_1 \equiv 0 \pmod 3$ or $n_1 \equiv 0 \pmod 3$. Similarly, $p_3 \times n_3 \equiv 10p+7 \equiv 1 \cdot p + 1 \equiv 1 \cdot 2 + 1 \equiv 0 \pmod 3$. It is either $p_3 \equiv 0 \pmod 3$ or $n_3 \equiv 0 \pmod 3$. Since p_1 and p_3 are the smallest odd prime factors, $p_3 = p_1 = 3$. $q_3 - q_1 = 6 = p_3 \times n_3 - p_1 \times n_1 = 3(n_3 - n_1)$. Then $n_3 - n_1 = 2$.

We then prove "only if" [See reviewer's comment (4)]. When $p_3 = p_1 = 3$ and $n_3 - n_1 = 2$, $p_3 \times n_3 - p_1 \times n_1 = 3(n_3 - n_1) = 3(2) = 6$. □

Since alone prime numbers are tail prime numbers, they also satisfy Theorem 6. Some tail prime numbers with their factorizations are shown below in Table 6.

Tail prime number	Number generated by SOUIPMT	Smallest prime factor	Other factor
$53 \equiv 2 \pmod 3$	**531**	**3**	**177**
	533	13	41
	537	**3**	**179**
	539	7	77
$89 \equiv 2 \pmod 3)^+$	**891**	**3**	**297**
	893	19	47
	897	**3**	**299**
	899	29	31
$107 \equiv 2 \pmod 3)^+$	**1071**	**3**	**357**
	1073	29	37
	1077	**3**	**359**
	1079	13	83
$113 \equiv 2 \pmod 3$	**1131**	**3**	**377**
	1133	11	103
	1137	**3**	**379**
	1139	17	67
$167 \equiv 2 \pmod 3)^+$	**1671**	**3**	**557**
	1673	7	239
	1677	**3**	**559**
	1679	23	73
$179 \equiv 2 \pmod 3$	**1791**	**3**	**597**
	1793	11	163
	1797	**3**	**599**
	1799	7	257
$251 \equiv 2 \pmod 3)^+$	**2511**	**3**	**837**
	2513	7	359
	2517	**3**	**839**
	2519	11	229

Table 6. For every alone prime number p with $p \equiv 2 \pmod 3), 10p + 7$ and $10p + 1$ have a common prime factor 3 and the difference of their remaining factors is 2. "+" means that the tail prime number is also an alone prime number.

Definition 7. Prime Number Tree Branch
Define prime number tree branch

$$[p; d_1 d_2 d_3 \ldots d_{n-1} d_n] \equiv \{p, \overline{pd_1}, \overline{pd_1 d_2}, \ldots, \overline{pd_1 d_2 \ldots d_{n-1}}, \overline{pd_1 d_2 \ldots d_{n-1} d_n}\},$$

where p is a fundamental prime number, d_i's are either 1, 3, 7 or 9. All set elements are prime.

In a prime number tree, prime number tree branches are prime number sequences starting with the same fundamental prime number but ending with different tail prime numbers. There are many branches in a prime number tree.

In Table 1, the prime number tree of 2 contains six branches, i.e.

[2;3333],

[2;3339],

[2;3399339],

[2;393],

[2;399333], and

[2;9399999].

2 is the fundamental prime number whereas 23333, 23339, 23399339, 2393, 2399333 and 29399999 all are tail prime numbers.

Definition 8. Length of Prime Number Tree *The length of a prime number tree is defined as the number of primes in the longest branch of the prime number tree. Let the longest branch be $[p; d_1 d_2 d_3 \ldots d_{n-1} d_n]$, where p is a fundamental prime number, and d_i's are either 1, 3, 7 or 9. The branch contains n+1 prime numbers. The length of the prime number tree is defined as $L = n + 1$.*

e.g. In the prime number tree of 2, the longest branches are [2;3399339] and [2;9399999]. They both contain 8 prime numbers. The length of the prime number tree of 2 is defined as 8.

Theorem 9. Classification of Prime Numbers by Prime Number Trees *Prime number trees give a complete classification of all prime numbers.*

Proof. p is a prime number. Suppose it is a fundamental prime number in the decimal number system, $p = \overline{d_1 d_2 d_3 \ldots d_n}$, where d_i's are either 0,1,2,3, ...,8, or 9, which are unit decimal integers, for $i = 1$ to n, and $d_i \neq 0$. The final digit d_n is either 1,3,7 or 9. Then p must produce a prime number tree by SOUIPMT and p is in it.

If p is not the fundamental prime number, there exists integers k and $f, k \geq f \geq 1$, $p = 10 \times \overline{d_1 d_2 d_3 \ldots d_f d_{f+1} \ldots d_k} + u$, where d_{f+1} to d_k and u are either 1, 3, 7 or 9, $\overline{d_1 d_2 d_3 \ldots d_f}$ is a fundamental prime number defined similar to the above. p is then belonged to the prime number tree of $\overline{d_1 d_2 d_3 \ldots d_f}$. Therefore, prime number trees contain and classify all prime numbers in the decimal number system. □

Theorem 10. Uniqueness of Prime Numbers in Prime Number Trees *Every prime number belongs to a unique prime number tree and prime number trees do not intersect.*

Proof. Suppose p is prime. If p is an alone prime number, then it is unique. If p is not an alone prime number, suppose it belongs to two different prime number trees. Let $p = 10 \times \overline{p_1 d_1 d_2 \ldots d_k} + u$ and $p = 10 \times \overline{p_2 e_1 e_2 \ldots e_m} + u$, where k and m are some positive integers, u, d_1 to d_k, e_1 to e_m are either 1,3,7, or 9, p_1 and p_2 are different fundamental prime numbers. Then

$$\begin{aligned} 0 &= p - p = (10 \times \overline{p_1 d_1 d_2 \ldots d_k} + u) - (10 \times \overline{p_2 e_1 e_2 \ldots e_m} + u), \\ 0 &= 10 \times (\overline{p_1 d_1 d_2 \ldots d_k} - \overline{p_2 e_1 e_2 \ldots e_m}), \\ \overline{p_1 d_1 d_2 \ldots d_k} &= \overline{p_2 e_1 e_2 \ldots e_m}. \end{aligned}$$

Contradiction! Therefore, $p_1 = p_2, m = k, d_i = e_i$ for $i = 1$ to k. Thus p belongs to a unique prime number tree. Since every prime number p belongs to a unique prime number tree, prime number trees do not intersect. □

Conjecture 11. Finite Length of Prime Number Trees *There is no prime number tree with infinite length.*

For every fundamental prime number $p_h = \overline{h_1 h_2 \ldots h_m}$ for some positive integer m, it is believed there exists a tail prime number $p_t = \overline{h_1 h_2 \ldots h_m d_1 d_2 d_3 \ldots d_n}$, for some positive integer n, generated by SOUIPMT, in the longest prime number tree branch, in every prime number tree such that

$$\begin{aligned} &10 \times \overline{h_1 h_2 \ldots h_m d_1 d_2 d_3 \ldots d_n} + 1, \\ &10 \times \overline{h_1 h_2 \ldots h_m d_1 d_2 d_3 \ldots d_n} + 3, \\ &10 \times \overline{h_1 h_2 \ldots h_m d_1 d_2 d_3 \ldots d_n} + 7 \text{ and} \\ &10 \times \overline{h_1 h_2 \ldots h_m d_1 d_2 d_3 \ldots d_n} + 9 \end{aligned}$$

all are composite. The length of the prime number tree is $n + 1$, which is believed to be finite.

Part B. Prime Number Trees in the Binary Number System

4. Binary Prime Number Trees

4.1. Binary Prime Number Trees

Let p be a prime number in the binary number system. e.g. $p = 2 = 10_{(2)}$. Similar to the prime number trees in the decimal number system shown in Part A, the prime number trees in the binary number system are also found and the length of the binary prime number trees can be proved to be finite. Alone binary prime number trees are significantly increased in amount.

Definition 12. Prime Number Tree in the Binary Number System *A prime number tree in the binary number system is a sequence of prime numbers. The sequence starts from a fundamental prime number. The following prime numbers are generated by attaching the digit 1 to the right hand side of the preceding prime number. The sequence ends at the tail prime number which cannot generate next prime number.*

The number generated by attaching the digit 0 to the right hand side of the preceding prime number must be even and non-prime so that this option is excluded.

e.g. $10_{(2)} = 2$, $101_{(2)} = 5$, $1011_{(2)} = 11$, $10111_{(2)} = 23$, $101111_{(2)} = 47$ are all prime but $1011111_{(2)} = 95$ is composite.

Some binary prime number trees are shown below.

(i) Prime number tree of $10_{(2)} = 2$

$$[10; 1111]_{(2)} = \{10_{(2)}, 101_{(2)}, 1011_{(2)}, 10111_{(2)}, 101111_{(2)}\}$$

(ii) Prime number tree of $11_{(2)} = 3$

$$[11; 1]_{(2)} = \{11_{(2)}, 111_{(2)}\}$$

(iii) Prime number tree of $1101_{(2)} = 13$

$$[1101;]_{(2)} = \{1101_{(2)}\} \text{ which is alone.}$$

Details of binary prime number trees of 2 to 97 are listed in the Table 7 below.

Prime Number in Binary Number System	Binary Prime Number Trees				
$10_{(2)} = 2$	$101_{(2)} = 5$	$1011_{(2)} = 11$	$10111_{(2)} = 23$	$101111_{(2)} = 47$	
$11_{(2)} = 3$	$111_{(2)} = 7$				
$101_{(2)} = 5$	*				
$111_{(2)} = 7$	*				
$1011_{(2)} = 11$	*				
$1101_{(2)} = 13$	(alone)				
$10001_{(2)} = 17$	(alone)				
$10011_{(2)} = 19$	(alone)				
$10111_{(2)} = 23$	*				
$11101_{(2)} = 29$	$111011_{(2)} = 59$				
$11111_{(2)} = 31$	(alone)				
$100101_{(2)} = 37$	(alone)				
$101001_{(2)} = 41$	$1010011_{(2)} = 83$	$10100111_{(2)} = 167$			
$101011_{(2)} = 43$	(alone)				
$101111_{(2)} = 47$	*				
$110101_{(2)} = 53$	$1101011_{(2)} = 107$				
$111011_{(2)} = 59$	*				
$111101_{(2)} = 61$	(alone)				
$1000011_{(2)} = 67$	(alone)				
$1000111_{(2)} = 71$	(alone)				
$1001001_{(2)} = 73$	(alone)				
$1001111_{(2)} = 79$	(alone)				
$1010011_{(2)} = 83$	*				
$1011001_{(2)} = 89$	$10110011_{(2)} = 179$	$101100111_{(2)} = 359$	$1011001111_{(2)} = 719$	$10110011111_{(2)} = 1439$	$101100111111_{(2)} = 2879$
$1100001_{(2)} = 97$	(alone)				

Table 7. Binary prime number trees of 2 to 97. "(alone)" means it is an alone binary prime number tree. "*" means it is included in a preceding binary prime number tree.

4.2. Finite Length of Binary Prime Number Trees

The length of decimal prime number trees is not yet known to be finite or infinite. It is probably to be finite by inspecting the prime number trees we worked out, even the amount of prime number trees is still a small number. However, in the binary number system we can prove that the length of binary prime number trees to be finite easily.

Theorem 13. (Fermat Little Theorem) *Let p be a prime, and let a be any number with $a \not\equiv 0 \pmod p$, then $a^{p-1} \equiv 1 \pmod p$.*

We skipped the proof since it is a well-known theorem and it can be found easily in textbooks such as J. H. Silverman's book [3].

Theorem 14. (Theorem of Finite Length of Binary Prime Number Trees) *The length of every binary prime number tree is finite.*

Proof. Let p be a fundamental prime number in the binary number system. Construct a binary number q by attaching $(p-1)$ digits of 1 to the right hand side of p. i.e.

$$\begin{aligned} q &= \overline{p\underbrace{111\ldots11}_{(p-1)1s}}_{(2)}, \\ q &= p \times 2^{p-1} + (2^{p-2} + 2^{p-3} + \cdots + 2 + 1) = p \times 2^{p-1} + \frac{2^{p-1}-1}{2-1}, \\ q &= p \times 2^{p-1} + 2^{p-1} - 1. \end{aligned}$$

By Theorem 13 (Fermat Little Theorem), $2^{p-1} \equiv 1 \pmod{p}$. Let $2^{p-1} = p \cdot h$ for some positive integer h. Then

$$q = p \times 2^{p-1} + p \cdot h = p(2^{p-1} + h).$$

q is divisible by p. Hence q is composite. According to Definition 8 of the length of a prime number tree, let L be the length of the binary prime number tree of q, $L \le p$. Then the length of the binary prime number tree is finite. This applies to every binary prime number tree. □

4.3. Prime Number Trees in the Ternary Number System

The first few prime numbers in the ternary number system is

$$2 = 2_{(3)}, 3 = 10_{(3)}, 5 = 12_{(3)}, 7 = 21_{(3)}, \ldots$$

We are interested in the length of prime number trees in the ternary number system. Cases of ternary prime number trees are shown below.

Case (I) $p = 2_{(3)} = 2$.

$20_{(3)}$ and $22_{(3)}$ are even. $21_{(3)} = 7$ is prime, which is included in Case (III).

Case (II) $p = 10_{(3)} = 3$.

$100_{(3)}, 101_{(3)}$ are composite, $102_{(3)} = 11$ is prime, which is included in Case (III).

Case (III) $p \ge 12_{(3)} = 5$.

In this case, $p \ge 12_{(3)} = 5$ and p is odd.

$\overline{p0}_{(3)} = 3 \times p$ is composite. $\overline{p1}_{(3)} = 3 \times p + 1$ is even and then composite.

$\overline{p2}_{(3)} = 3 \times p + 2$ is odd. Construct a ternary number q by attaching $(p-1)$ digits of 2 to the right hand side of p. Note that such a ternary number q is always odd. We then have to apply Fermat Little Theorem to find out its odd-even parity.

$$\begin{aligned} q =& \overline{p\underbrace{22\ldots22}_{(p-1)2s}}_{(3)} \\ =& p \times 3^{p-1} + (3^{p-2} + 3^{p-3} + \cdots + 3 + 1) \\ =& p \times 3^{p-1} + \frac{3^{p-1}-1}{3-1}. \end{aligned}$$

By Fermat Little Theorem, $3^{p-1} \equiv 1 \pmod{p}$ for $p \geq 5$. Let $3^{p-1} - 1 = p$, for some positive integer w. Since $3^{p-1} - 1$ is even and p is odd, w is even.

$$q = \overline{p\underbrace{22\ldots22}_{(p-1)2s}}_{(3)} = p \times 3^{p-1} + \frac{1}{2}(p \cdot w),$$

which is divisible by p and then q is composite.

Hence, in this case, a ternary prime number tree starting from a fundamental prime number $p \geq 12_{(3)} = 5$ must be finite in length.

From cases (I), (II) and (III), we conclude that the length of every ternary prime number tree is finite.

We hope the various concepts in prime number trees can give some insights to the understanding of prime numbers.

REFERENCES

[1] The On-Line Encyclopedia of Integer Sequences, *Erdős-Selfridge classification of primes*, http://oeis.org/wiki/Erd%C5%91s-Selfridge_classification_of_primes .
[2] G.H. Hardy and E.M. Wright, *An Introduction to the Theory of Numbers*, 4th edition, Oxford University Press, Oxford, 1960.
[3] Joseph H. Silverman, *A Friendly Introduction to Number Theory*, 3rd Edition, Pearson Prentice Hall, 2005.

APPENDIX A

List of Decimal Prime Number Trees

Prime Numbers	Decimal Prime Number Trees							
2	23	233	2333	233333				
				23339				
			2339	23399	233993	2339933	23399339	
		239	2393					
			2399	23993	239933	2399333		
	29	293	2939	29399	293999	2939999	29399999	
3	31	311	3119	31193				
		313	3137	31379				
		317						
	37	373	3733	37337	373379	3733799	37337999	
				37339	373393			
			3739	37397				
		379	3793					
			3797					
5	53							
	59	593	5939	59393	593933	5939333	59393339	
				59399	593993			
		599						
7	71	719	7193	71933	719333			
	73	733	7331					
			7333	73331				
		739	7393	73939	739391	7391913	73939133	
					739393	7393931		
						7393933		
					739397			
					739399			
	79	797						
11	113							
13	131	1319						
	137	1373						
	139	1399	13997					
			13999	139991	1399913	13999133		
					1399919			
				139999	1399999			
17	173	1733	17333					
	179							
19	191	1913	19139					
	193	1931	19319					
		1933	19333	193337				
	197	1973	19739					
		1979	19793	197933	1979339	19793393	197933933	1979339333
								1979339339
	199	1993	19937	199373				
				199379				
		1997	19973	199739				
		1999	19991					
			19993	199931				
				199933	1999331	19993319		
					1999339			
			19997					
23	*							
29	*							
31	*							
37	*							
41	419							
43	431							
	433	4337						
		4339	43391					
			43397					

			43399						
	439	4391	43913	439133	4391339				
		4397	43973						
47	479	4793	47933						
			47939						
		4799							
53	*								
59	*								
61	613	6131							
		6133	61331						
			61333	613337	6133373				
			61339						
	617	6173							
	619	6197	61979	619793					
		6199	61991						
67	673	6733	67339	673391					
				673397					
				673399	6733997				
		6737							
	677	6779							
71	*								
73	*								
79	*								
83	839								
89	(alone)								
97	971	9719							
	977								
101	1013	10133	101333						
	1033	10333	103333	1033337					
				1033339	10333391				
		1037							
	1039	10391	103913	1039139					
			103919						
		10399	103991						
			103993	1039931					
			103997	1039979	10399793	103997939	1039979393		
							1039979399	10399793993	103997939939
107	(alone)								
109	1091								
	1093	10937	109379						
		10939	109391						
			109397						
	1097	10973							
		10979	109793	1097933					
133	*								
127	1277								
	1279	12791	127913	1279133	1279133				
		12799	127997						
131	*								
137	*								
139	*								
149	1493	14939	149393						
			149399						
	1499								
151	1511								
157	1571								
	1579	15791							
		15797							
163	1637								
167	(alone)								
173	*								
179	*								
181	1811	18119	181193	1811939					
			181199	1811993					
191	*								
193	*								
197	*								
199	*								
211	2111								
	2113	21139							
223	2237								
	2239	22391	223919						
		22397							
227	2273	22739	227393						

			227399	2273993					
229	2293	22937	229373						
	2297	22973	229739						
233	*								
239	*								
241	2411	24113							
	2417	24179	241793	2417939	24179399				
251	(alone)								
257	2579	25793	25793						
			25799	257993	2579939				
263	2633	26339	263399						
269	2693								
	2699	26993	269939	2699393					
271	2711								
	2713								
	2719	27191	271919	2719193	27191939				
		27197							
277	2777	27773							
		27779	277793						
281	2819								
283	2833								
	2837								
293	*								
307	3079								
311	*								
313	*								
317	(alone)								

Table A1. Decimal prime number trees. "(alone)" means it is an alone prime number. "*" means it is included in a preceding decimal prime number tree.

Reviewer's Comments

The presentation of this paper is very good. The following is a list of corrections and stylistic suggestions.

1 The reviewer has comments on the wordings, which have been amended in this paper.
2 "The solution to the equation is if and only if" should be rewritten as "The equation has a unique solution".
3 "We prove "if" first" should be deleted.
4 "We then prove "only if"." should be deleted.