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ROLLING WITHOUT SLIDING

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Abstract. First, we define three kinds of measures of angles in order to handle rolling and rotation.
In Part One, we give the relations among them:

$$d\theta = d\varphi - d\phi$$

And the accumulated exterior angles φ and ϕ are highly depended on the geometric features of the objects concerned (i.e. curvatures of the boundaries of objects):

$$dD = R(\varphi)d\varphi \text{ and } dD = -r(\phi)d\phi.$$

In Part Two, we further study the length of the trajectory formed by certain point on the rolling object. The result is:

$$dT_p = L_p(\phi)|d\theta|.$$

Also we will give some applications of these results under the circumstance that the curvatures are uniform (The curvatures can be positive, zero and negative).

1. Introduction

In this report, we want to handle the relationship between the distance traveled (D) and the angle of self-rotation (θ).

What would D be when we have been given θ? Obviously,

$$D = r\theta,$$

where r is the radius of rolling circle. However, that is only a special case.

When we consider a one-dollar coin rolling on another one-dollar coin, we can get an interesting result.

[1] This work is done under the supervision of the authors' teacher, Mr. Hon-Wai Yung.

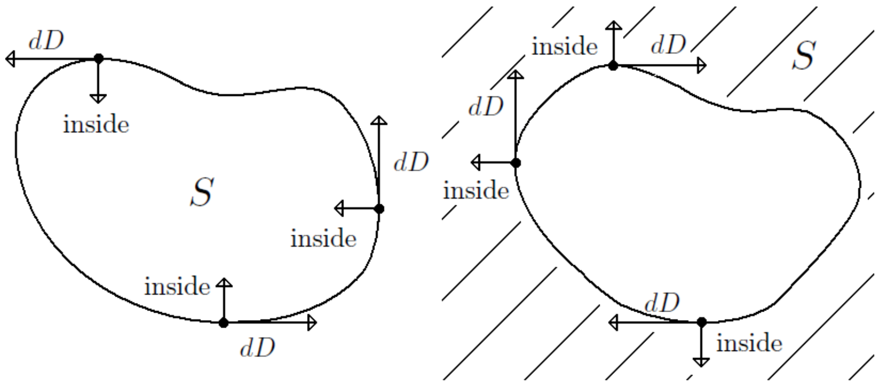
When $D = \pi r$, $\theta = 2\pi$. The value of θ is twice of we expected. Do you know why?

In order to solve this problem, in our report, we use S to represent the stationary object and M to represent the rolling object.

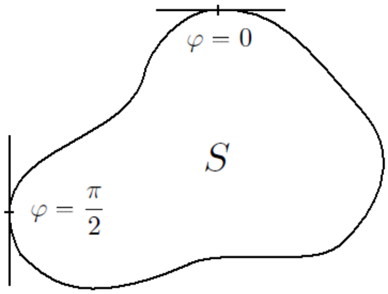
Throughout our report, we assume the rolling is **Rolling Without Sliding**. Rolling without sliding means that when M is rolling on S, the instantaneous velocity of the contact point of M on S is zero.

Also we assume that M is rolling outside of S, conversely, S is "rolling" outside of M.

D is defined as the directed distance that M has traveled when M is rolling on S without sliding. dD is positive when the inside of S is always on the left hand side of dD.

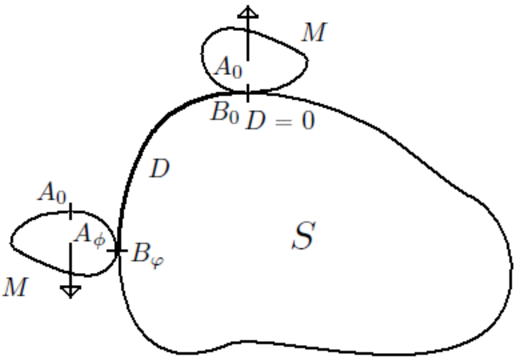


φ is defined as inclination of the tangent at every boundary point of S, i.e. the accumulated exterior angle of S. **It gives the parameterized form (but not unique) of the points on boundary of S.** We use B_φ to represent corresponding boundary point of S.



Similarly, ϕ is defined as the accumulated exterior angle of M, that is, **it gives the parameterized form (but not unique) of the points on boundary of M. We use A_ϕ to represent corresponding boundary point of M.**

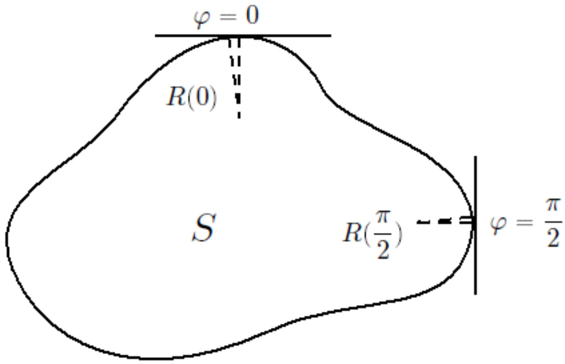
Throughout this report, we assume that the rolling starts when A_0 of M touches B_0 of S, that is, When $D = 0$, φ and ϕ are both zero.



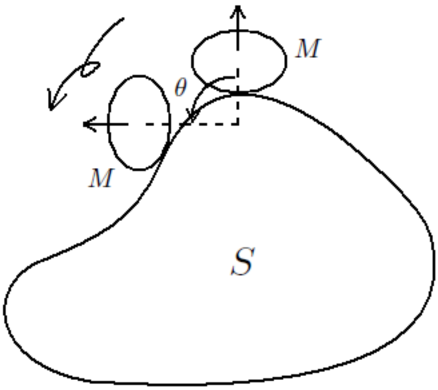
Comparing with the **length parametrization** l of curve in usual practice, we use **exterior-angle parametrization** φ and ϕ.

$R(\varphi)$ is defined as the radius of curvature of S at B_φ.

$r(\phi)$ is defined as the radius of curvature of M at A_ϕ.



Assume there is an arrow stick to M. When M is rolling about S, the angle that measures the change of the direction of that arrow is defined as θ. Actually, θ measures the self-rotation of M during rolling.



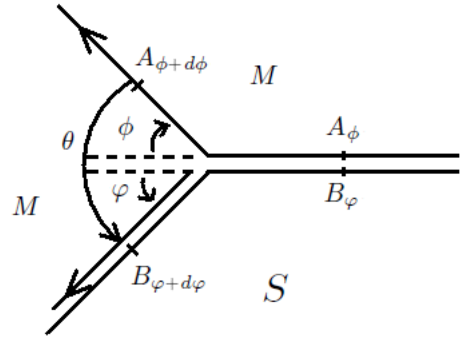
In Part One, we will give the relationship among these variables.

2. Main Body(Part 1)

Lemma 1. *Suppose Object M is rolling without sliding on Object S, then*

$$d\theta = d\varphi - d\phi.$$

Proof. First, A_0 of M touches B_0 of S. After rolling, $A_{\phi+d\phi}$ of M touches $B_{\varphi+d\varphi}$ of S. Without loss of generality, we assume the arrow on M is tangent to $A_{\phi+d\phi}$.



Refer to the diagram, we have,

$$\begin{aligned} d_\theta &= (-d\phi) + d\varphi \\ &= d\varphi - d\phi \end{aligned}$$ □

Lemma 2. *Suppose Object M is rolling without sliding on Object S, then*

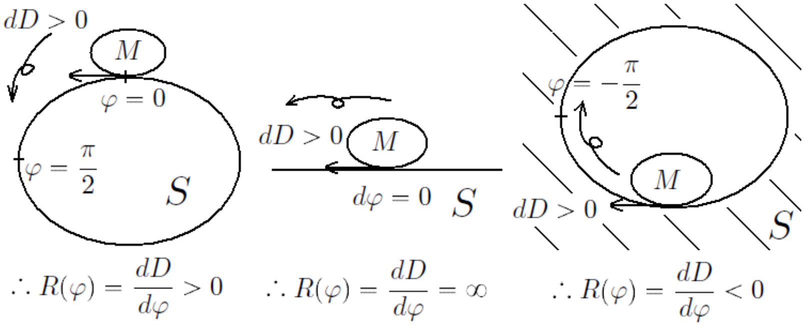
$$dD = R(\varphi)d\varphi.$$

Proof. In Differential Geometry, we have,

$$\kappa(\varphi) = \frac{d\varphi}{dD},$$

where $\kappa(\varphi)$ is the curvature of the boundary of S at B_φ.
Since $R(\varphi) = \dfrac{1}{\kappa(\varphi)}$, $dD = R(\varphi)d\varphi$. □

Notice that $R(\varphi)$ may not necessarily be positive nor finite in some cases.



Obviously, for a circle, the radius of curvature is identical to its radius (or the negative of its radius), we have the following corollary.

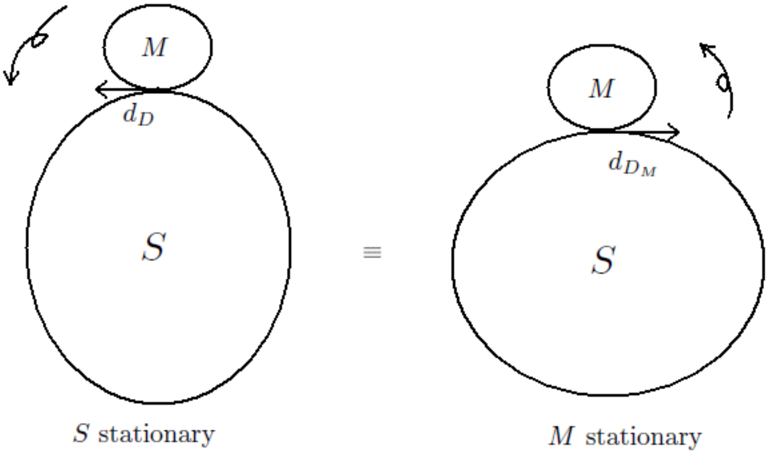
Corollary 3. *Suppose Object S is a disk of radius R,*

$$D = R\varphi.$$

Lemma 4. *Suppose Object M is rolling without sliding on Object S, then*

$$dD = -r(\phi)d\phi.$$

Proof. Relatively speaking, we may set M be the "stationary" one. Then we define D_M as the distance of S traveled when S is "rolling" on M.



By Lemma 2, $dD_M = r(\phi)d\phi$ and by the diagram, $dD_M = -dD$. The result follows. $\square$

Theorem 5. *Suppose M and S are circles of radius r and R. If M is rolling outside of S, then*

$$D = \frac{Rr}{R+r}\theta.$$

Proof. Notice that $R(\varphi) \equiv R$ and $r(\phi) \equiv r$, and from Lemma 2 and Lemma 4, we have:

$$D = R\varphi \text{ and } D = -r\phi.$$

Substitute them into $\theta = \varphi - \phi$ (by Lemma 1),

$$\theta = \frac{D}{R} - \frac{D}{-r}$$
$$D = \frac{Rr}{R+r}\theta.$$ □

Therefore, if $R = r$ (i.e., the case in the introduction, a one-dollar coin rolls about another one-dollar coin),

$$D = \frac{1}{2}r\theta.$$

Theorem 6. *Suppose M and S are circles of radius r and R, where $r < R$. If M is rolling inside of S, then*

$$D = \frac{Rr}{R-r}\theta.$$

Proof. Notice that $R(\varphi) \equiv -R$ and $r(\phi) \equiv r$, and from Lemma 2 and Lemma 4, we have:

$$D = -R\varphi \text{ and } D = -r\phi.$$

Substitute them into Lemma 1:

$$\theta = \frac{D}{-R} - \frac{D}{-r}$$
$$D = \frac{Rr}{R-r}\theta.$$

Actually, you can solve all problems about rolling without sliding by using Lemma 1, Lemma 2 and Lemma 4 under some boundary conditions.

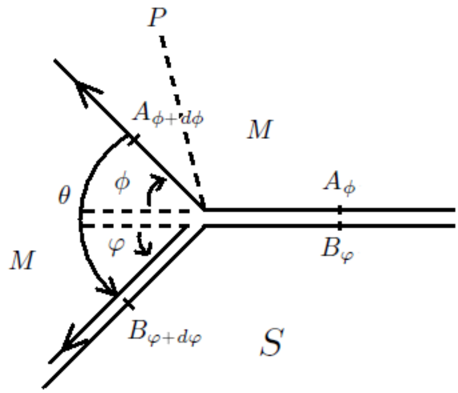
3. Main Body (Part 2)

In part 1, we have considered the distance that M travels when M is rolling on S. Now in part 2, we draw our attention on the distance that a certain point P at M travels, that is, the length of the trajectory of P. T_P is defined as the length of the trajectory that P has traveled when M is rolling on S without sliding. In order to solve this problem, we define a new function $L_P(\phi)$, that is the distance between P and A_ϕ.

Lemma 7. *Suppose Object M is rolling without sliding on Object S, then*

$$dT_P = L_P(\phi)|d\theta|.$$

Proof. Refer to the following diagram,



The result follows. □

Theorem 8. *Suppose P is a point on the boundary of circle M with radius r, while Object S is a straight line. Then the length of the trajectory of P when M rolls one cycle is $8r$.*

Proof. Since $d\varphi = 0$, $d\theta = -d\phi$. M rolls one cycle, that is, θ changes from 0 to 2π, or equivalently, ϕ changes from 0 to -2π.

Without loss of generality, suppose $A_0 = P$.

By Cosine Law,

$$\begin{aligned} L_P(\phi) &= \sqrt{r^2 + r^2 - 2r^2 \cos\phi} \\ &= r\sqrt{2(1-\cos\phi)} \\ &= r\sqrt{2(2\sin^2\frac{\phi}{2})} \\ &= 2r\left|\sin\frac{\phi}{2}\right|. \end{aligned}$$

Then, the required length T_P can be found by:

$$\begin{aligned} T_P &= \int_{\theta\in[0,2\pi]} 2r\left|\sin\frac{\phi}{2}\right| |d\theta| \\ &= \int_0^{2\pi} 2r\sin\frac{\theta}{2} \\ &= 2r\left[-2\cos\frac{\theta}{2}\right]_0^{2\pi} \\ &= 2r[-2(-1-1)] \\ &= 8r. \end{aligned}$$

□

Actually, this trajectory is well known as cycloid. In many calculus books, its length can be found by first giving the parametric equation of cycloid and then do the integration. However, in the proof of Theorem 8, you may use another more general method to do so. Further, if object S is no longer a straight line, e.g. a circle, we can still find out the length of the "general" cycloid.

Theorem 9. *Suppose P is a point on the boundary of circle M with radius r, while Object S is a circle with radius R where $R = r$. Then the length of the trajectory of P when M revolves round S one cycle is $16r$.*

Proof. By Lemma 2 and Lemma 4,

$$R(\varphi)d\varphi = -r(\phi)d\phi.$$

Since $R(\varphi) \equiv R$ and $r(\phi) \equiv r = R$, we have

$$d\varphi = -d\varphi.$$

Then by Lemma 1,

$$d\theta = d\varphi - d\phi = -2d\phi$$

M revolves round S one cycle, that is, φ changes from 0 to 2π , or equivalently, ϕ changes from 0 to -2π. The required length T_P can be found by:

$$
\begin{aligned}
T_P &= \int_{\varphi\in[0,2\pi]} L_P(\phi)|d\theta| \\
&= \int_0^{-2\pi} L_P(\phi)(-2d\phi) \\
&= -2\int_0^{-2\pi} 2r\left|\sin\frac{\phi}{2}\right| d\phi \\
&= 2\int_0^{-2\pi} 2r\sin\frac{\phi}{2}d\phi \quad (\text{since } \sin\frac{\phi}{2}\leqslant 0 \text{ when } -2\pi\leqslant\phi\leqslant 0) \\
&= 4r\left[-2\cos\phi 2\right]_0^{-2\pi} \\
&= 4r[-2(-1-1)] \\
&= 16r.
\end{aligned}
$$

□

Theorem 10. *Suppose P is a point on the boundary of circle M with radius r, while Object S is a circle with radius R where $R = 2r$. Then the length of the trajectory of P when M revolves round one cycle inside S is $8r$.*

Proof. By Lemma 2 and Lemma 4,

$$R(\varphi)d\varphi = -r(\phi)d\phi.$$

Since $R(\varphi)\equiv -R$ and $r(\phi)\equiv r=\dfrac{R}{2}$, we have

$$d\varphi = \frac{d\varphi}{2}.$$

Then by Lemma 1,

$$d\theta = d\varphi - d\phi = -\frac{1}{2}d\phi.$$

M revolves round S one cycle, that is, φ changes from 0 to 2π , or equivalently, ϕ changes from 0 to 4π. The required length T_P can be found by:

$$\begin{aligned}
T_P &= \int_{\varphi\in[0,2\pi]} L_P(\phi)|d\theta| \\
&= \int_0^{4\pi} L_P(\phi)\left|-\frac{1}{2}d\phi\right| \\
&= -\frac{1}{2}\int_0^{4\pi} 2r\left|\sin\frac{\phi}{2}\right| d\phi \\
&= r\left(\int_0^{2\pi}\sin\frac{\phi}{2}d\phi - \int_{2\pi}^{4\pi}\sin\frac{\phi}{2}d\phi\right) \\
&= r\left(\left[-2\cos\frac{\phi}{2}\right]_0^{-2\pi} - \left[-2\cos\frac{\phi}{2}\right]_{2\pi}^{4\pi}\right) \\
&= -2r[(-1-1)-(1+1)] \\
&= 8r.
\end{aligned}$$ □

Notice that the length of the trajectory of P is just the double of the diameter of S, that means the trajectory is just along the straight line B_0B_π and $B_\pi B_{2\pi}$.

Up to now, the trajectories in all the cases we have considered are completed (the starting position is at which the ending position is) when M revolves around S one cycle. That is, M starts rolling with contact points A_0 and B_0 and then ends with contact points $A_{2n\pi}$ and $B_{2\pi}$, where n is some integer. We call it complete trajectory in 1-cycle.

Combining Lemma 2 and Lemma 4, we have:

$$R(\varphi)d\varphi = -r(\phi)d\phi$$

If we only consider two circles, we have $R(\varphi) \equiv R$ and $r(\phi) \equiv r$.

$$R\varphi = -r\phi \tag{1}$$

Since Object M is rolling one cycle outside object S, φ changes from 0 to 2π, i.e. $\varphi = 2\pi$.

And for a complete trajectory (start at A_0 and end at A_0), ϕ changes from 0 to $2n\pi$, where n is some positive integer, i.e. $\phi = -2n\pi$. Therefore,

$$\begin{aligned}
R(2\pi) &= -r(-2n\pi) \\
R &= nr
\end{aligned}$$

That is the condition of complete trajectory in 1-cycle.

Then, we can consider the conditions needed that the trajectory is completed after m cycles, we call it complete trajectory in m-cycle.

Similarly, we have $\varphi = 2m\pi$ and $\phi = -2n\pi$ for some integer n. Therefore,

$$\begin{aligned} R(2m\pi) &= -r(-2n\pi) \\ R &= \frac{n}{m}r \end{aligned}$$

After knowing the ratio of r and R, we can find out the length of trajectory of point A_0 in m cycles. Substitute 1 into Lemma 1,

$$\begin{aligned} \theta &= \phi + \frac{r}{R}\phi \\ &= \left(1 + \frac{r}{\frac{n}{m}r}\right)\phi \\ &= \left(1 + \frac{m}{n}\right)\phi. \end{aligned}$$

From Lemma 2, we have

$$\begin{aligned} T_P &= \int_{\varphi\in[0,2m\pi]} L_P(\phi)|d\theta| \\ &= \int_0^{-2n\pi} 2r\left|\sin\frac{\phi}{2}\right|\left(-d\left(1+\frac{m}{n}\right)\phi\right) \\ &= -2r\left(1+\frac{m}{n}\right)\int_0^{-2n\pi}\left|\sin\frac{\phi}{2}\right|d\phi \\ &= 2r\left(1+\frac{m}{n}\right)n\int_0^{2\pi}\sin\frac{\phi}{2}d\phi \\ &= 2r(n+m)\left[-2\cos\frac{\phi}{2}\right]_0^{2\pi} \\ &= 2r(n+m)[-2(-1-1)] \\ &= 8r(n+m). \end{aligned}$$

Therefore, the length of this trajectory is $8r(n+m)$.

We are now going to investigate the conditions required for the trajectory to be completed when a circle is rolling inside another circle in m-cycle.

If we only consider two circles, we have $R(\varphi) \equiv -R$ and $r(\phi) \equiv r$.

$$R\varphi = r\phi \tag{2}$$

Similarly, we have: $\varphi = 2m\pi$ and $\phi = 2n\pi$ for some integer n. Therefore,

$$\begin{aligned} R(2m\pi) &= r(2n\pi) \\ R &= \frac{n}{m} r \quad \text{(note that } n > m). \end{aligned}$$

After knowing the ratio of r and R, we can find out the length of trajectory of point A_0 in m cycles. Substitute 2 into Lemma 1,

$$\begin{aligned} \theta &= \phi - \frac{r}{R}\phi \\ &= \left(1 - \frac{r}{\frac{n}{m}r}\right)\phi \\ &= \left(1 - \frac{m}{n}\right)\phi. \end{aligned}$$

From Lemma 2, we have

$$\begin{aligned} T_P &= \int_{\varphi \in [0, 2m\pi]} L_P(\phi)|d\theta| \\ &= \int_0^{-2n\pi} 2r \left|\sin\frac{\phi}{2}\right| \left(-d\left(1 - \frac{m}{n}\right)\phi\right) \\ &= -2r\left(1 - \frac{m}{n}\right)\int_0^{-2n\pi} \left|\sin\frac{\phi}{2}\right| d\phi \\ &= 2r\left(1 - \frac{m}{n}\right) n \int_0^{2\pi} \sin\frac{\phi}{2} d\phi \\ &= 2r(n-m)\left[-2\cos\frac{\phi}{2}\right]_0^{2\pi} \\ &= 2r(n-m)[-2(-1-1)] \\ &= 8r(n-m). \end{aligned}$$

Therefore, the length of this trajectory is $8r(n-m)$.

4. Conclusion

Working very hard for this report, we would like to solve the problems related to the trajectory produced by various sizes of circles rolling on different objects. We think it may be one of the most interesting and absorbing geometric problems in mathematics.

Through the successes and failures that we encountered in the past few months, we attained a lot of wonderful results. Moreover, all of us learnt a lot more than we expected. We then made uses of the results to establish many theorems.

Appendix A.

If M or i are Polygons, we will find that some boundary points (more than one) will have the same accumulated exterior angles; and some boundary points will have more than one accumulated exterior angles.

For example, all points on the one side of the square are A_0, $A_{\frac{\pi}{2}}$, A_π or $A_{\frac{3\pi}{2}}$, etc respectively. And the extreme points of the square are $A_{\left[0,\frac{\pi}{2}\right]}$, $A_{\left[\frac{\pi}{2},\pi\right]}$, $A_{\left[\pi,\frac{3\pi}{2}\right]}$, $A_{\left[\frac{3\pi}{2},2\pi\right]}$ etc respectively.

Clearly the exterior angle at $A_{[\phi_1,\phi_2]}$ must be $\phi_2 - \phi_1$. Therefore,

$$\Delta T_P = A_{[\phi_1,\phi_2]}P|\phi_2 - \phi_1|.$$

Theorem 11. *Suppose M is a regular n-polygon inscribed in a circle of radius r. Suppose P is any extreme point of M. S is a straight line. Then the length of the trajectory of P when M rolls one cycle is*

$$\frac{4\pi r\left(1+\cos\dfrac{\pi}{n}\right)}{n\sin\dfrac{\pi}{n}}.$$

Reviewer's Comments

The reviewer has only comments on the wordings, which have been amended in this paper.